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Newton's method for a rational matrix equation occurring in stochastic control

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Abstract

We study a general class of rational matrix equations, which contains the continuous (CARE) and discrete (DARE) algebraic Riccati equations as special cases. Equations of this type were encountered in [SIAM J. Control and Optimization 36 (1998) 1504–1538; Stochastics and Stochastics Reports, 65 (1999) 255–297], where H^∞-type problems of disturbance attenuation for stochastic linear systems were studied. We develop a unifying framework for the analysis of these equations based on the theory of (resolvent) positive operators and show that they can be solved by Newton's method starting at an arbitrary stabilizing matrix. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

A substantial branch of optimal linear control theory is concerned with the minimization of quadratic functionals (in the state vector x and the input vector u) constrained by a linear system. The solutions of these problems lead to certain matrix equations and inequalities (for a Hermitian matrix X) that differ depending both on the type of the quadratic objective functional—which might be semidefinite or indefinite—and on the type of linear system equation—which can be continuous

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or discrete, and in both cases might incorporate state-dependent or input-dependent multiplicative noise or both.

Positive semidefinite functionals (definite in u) are regarded in the classical LQ-control theory. In the deterministic case they lead to the classical *definite* continuous and discrete algebraic Riccati equations (e.g. [15]). We call them definite, because the constant term in the corresponding linear matrix inequality is positive (semi)definite. The greatest solutions of the latter yield the optimal feedback gain matrix and thus the optimal control. These matrix equations are solvable under generic stabilizability conditions. The incorporation of multiplicative noise in the LQ problem leads to more general definite matrix equations, which we call *stochastic algebraic Riccati equations* (STARE). They comprise the previous ones as special cases but contain additional terms, that are monotonic in X (with respect to the partial ordering of Hermitian matrices), and which stem from the diffusion part of the system (see [23,24]). Like their deterministic counterparts, the definite stochastic equations can be solved under certain stabilizability conditions.

Indefinite functionals (negative semidefinite in x, positive definite in u) are regarded in the disturbance attenuation problem, which in the deterministic case is widely known as H^∞-control theory. A centerpiece of this theory is the so-called bounded real lemma that characterizes the input–output norm, or attenuation value of the given system via the solvability of certain matrix inequalities. The corresponding matrix equations are the *indefinite* counterparts of the matrix equations from LQ-control theory, depending on whether the underlying system is continuous or discrete and whether it incorporates multiplicative noise or not. But the solvability question is more difficult for the indefinite matrix equations. These equations are of the following general form:

$$
\begin{aligned}
A^*X + XA + \sum_{i=1}^{N} {A_0^i}^* X A_0^i + P_0 - \left(XB + \sum_{i=1}^{N} {A_0^i}^* X B_0^i + S_0 \right) \\
\times \left(\sum_{i=1}^{N} {B_0^i}^* X B_0^i + Q_0 \right)^{-1} \left(XB + \sum_{i=1}^{N} {A_0^i}^* X B_0^i + S_0 \right)^* = 0. \qquad (1)
\end{aligned}
$$

In the deterministic case (where the sums are absent, see Section 5), there exist various approaches to the solution of definite and indefinite Riccati type matrix equations (see e.g. [5,15,16,19,21] and references therein):

- Firstly, the solvability can be characterized by so-called frequency domain criteria, which involve the transfer function of the system and which are trivially satisfied in the definite case. If the frequency condition is satisfied, then the matrix equation can basically be solved by a spectral factorization of the transfer function.

- Secondly, the solvability can be characterized by analyzing the spectrum of the associated Hamiltonian matrix, and solutions can be constructed by computing invariant Lagrangian subspaces of the Hamiltonian.
- Thirdly, the greatest solution of the matrix equations can be computed iteratively, e.g. by a Newton–Kantorovich procedure. Roughly speaking, the iteration starts with an arbitrary stabilizing matrix and it converges to the greatest solution of the equation, if the equation is solvable at all.

These approaches apply to both continuous and discrete equations in a similar way, and in many cases it is possible to convert a result for a continuous equation to a result for the discrete equation by considering an appropriate linear fractional transformation. Nevertheless, both cases usually need to be treated individually.

So far there has been no success in applying the first two methods in the stochastic case, since here neither a transfer function nor a Hamiltonian are available. This is partly due to the fact, that the fundamental solution of a stochastic differential equation can in general not be given in closed form.

But the iterative method has been applied successfully to the definite stochastic equation (with $B_0^i = 0$, $Q_0 > 0$, $S_0 = 0$, $P_0 \geqslant 0$, and (A, P_0) detectable in (1)) by Wonham in [24], for later extensions, see e.g. in [6,10]. Wonham basically applies a version of Newton's algorithm; his proof of convergence relies in a central point on the fact that the given equation is definite.

The indefinite stochastic algebraic Riccati equation was first encountered in [8,9,12], where the disturbance attenuation problem was studied for continuous and discrete linear systems with both state- and input-dependent multiplicative white noise.

None of the above results applies directly to the solution of this equation, but an iterative procedure seems to be the most promising approach. It is the task of this paper to present such a procedure. Actually, we dispense with all definiteness assumptions and consider an equation that comprises all the equations mentioned above as special cases. Thus a unifying approach is given. We proceed as follows:

In Section 2, we give a brief account of a stochastic version of the bounded real lemma and derive a special version of the rational matrix operator to be studied in the sequel. Sections 3 and 4 are devoted to the study of *resolvent positive* operators and *concave* operators, which play a central role in our discussion. In Section 5, we introduce a general rational matrix equation and show that the corresponding rational matrix operator is concave with resolvent positive derivative. These properties are used in Section 6 to establish a non-local convergence result of the Newton-iteration to solve the rational matrix equation. In the latter sections, we draw some conclusions from our main result concerning the structure of the set of solutions, the speed of convergence and the dependence of the greatest solution on the data.

2. A Riccati type rational matrix operator

Regard the linear Itô differential equation

$$\begin{aligned} \mathrm{d}x(t) &= Ax(t)\,\mathrm{d}t + Bv(t)\,\mathrm{d}t + \sum_{i=1}^{N} A_0^i x(t)\,\mathrm{d}w_i(t) + \sum_{i=1}^{N} B_0^i v(t)\,\mathrm{d}w_i(t), \\ z(t) &= Cx(t) + Dv(t) \end{aligned} \tag{2}$$

where $(A, C) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n}$, and

$$(A_0^i, B_0^i, B, D) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times \ell} \times \mathbb{K}^{n\times \ell} \times \mathbb{K}^{q\times \ell} \quad (\mathbb{K} = \mathbb{R} \text{ or } \mathbb{K} = \mathbb{C}).$$

The $(w_i(t))_{t\in\mathbb{R}_+}$ are independent zero mean real Wiener processes on a probability space $(\Omega, \mathscr{F}, \mu)$ with respect to an increasing family $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ of σ-algebras $\mathscr{F}_t \subset \mathscr{F}$.

Let $L^2_w(\mathbb{R}_+, \mathbb{K}^\ell)$ denote the corresponding space of non-anticipating stochastic processes v with values in $\mathbb{K}^\ell$ and norm

$$|v(\cdot)|^2_{L^2_w} := \mathscr{E}\left(\int_0^\infty |v(t)|^2\,\mathrm{d}t\right) < \infty,$$

where $\mathscr{E}$ denotes expectation.

It is known from Itô-theory, that for all $(x_0, v) \in \mathbb{K}^n \times L^2_w(\mathbb{R}_+, \mathbb{K}^\ell)$, there exists a unique solution $x(\cdot, x_0, v)$ of (3) and thus also a unique output process $z(\cdot, x_0, v)$.

We write $z(\cdot, 0, v) = \boldsymbol{L}v(\cdot)$, and call $\boldsymbol{L}$ the *perturbation operator* of system (3). It describes the effect of the input process v (viewed as a stochastic disturbance) on the output process z (interpreted as the vector of the to be controlled variables).

Definition 2.1. System (3) is said to be *internally* (*exponentially mean square*) *stable* if

$$\exists M, \omega > 0 : \forall x_0 \in \mathbb{K}^n,\ t \geqslant 0 : \mathscr{E}|x(t)|^2 \leqslant M\mathrm{e}^{-\omega t}|x_0|^2,$$

where $x(\cdot) = x(\cdot, x_0, 0)$ is the solution of the unperturbed system (with $v(\cdot) \equiv 0$). System (3) is called *externally stable* if $\boldsymbol{L}$ is a bounded operator $\boldsymbol{L} : L^2_w(\mathbb{R}_+, \mathbb{K}^\ell) \to L^2_w(\mathbb{R}_+, \mathbb{K}^q)$.

In [8] it was shown, that internal stability of (3) implies external stability.

The norm $|\boldsymbol{L}|$ of the disturbance operator is of special interest. In the deterministic case (if all A_0^i, B_0^i vanish) it is equal to the H^∞-norm of the associated rational transfer matrix. Thus $|\boldsymbol{L}|$ can be seen as a generalized H^∞-type-norm for the stochastic system (3). In [12] a bounded real lemma for stochastic systems was proved.

Let $\mathscr{H}^n \subset \mathbb{K}^{n\times n}$ denote the real space of $n \times n$ Hermitian matrices with entries in $\mathbb{K}$, endowed with the usual ordering $X \leqslant Y$ of Hermitian matrices (see Section 3). For any $\gamma > 0$, we define the affine linear operators $P : \mathscr{H}^n \to \mathscr{H}^n$, $Q : \mathscr{H}^n \to \mathscr{H}^\ell$ and $S : \mathscr{H}^n \to \mathbb{K}^{n\times\ell}$ by

$$P(X) = A^*X + XA + \sum_{i=1}^{N} A_0^{i\,*} X A_0^i - C^*C,$$

$$S(X) = XB + \sum_{i=1}^{N} A_0^{i\,*} X B_0^i - C^*D,$$

$$Q(X) = \sum_{i=1}^{N} B_0^{i\,*} X B_0^i + \gamma^2 I - D^*D.$$

Consider the rational Riccati-type operator $\mathscr{R} : \operatorname{dom}\mathscr{R} \to \mathscr{H}^n$ given by

$$\begin{aligned} &\mathscr{R}(X) = P(X) - S(X)Q(X)^{-1}S(X)^*,\\ &X \in \operatorname{dom}\mathscr{R} = \{X \in \mathscr{H}^n \mid Q(X) > 0\}. \end{aligned} \tag{3}$$

Theorem 2.2 (Stochastic bounded real lemma [8]).

$$\begin{aligned} &\textit{System (3) is internally stable and } |\mathbf{L}| < \gamma\\ &\quad \Longleftrightarrow \exists X < 0 : Q(X) > 0 \textit{ and } \mathscr{R}(X) > 0. \end{aligned} \tag{4}$$

Since $Q(\cdot)$ is affine and monotone the domain of definition of $\mathscr{R}$ is convex and saturated above: $\operatorname{dom}\mathscr{R} + \mathscr{H}^n_+ \subset \operatorname{dom}\mathscr{R}$. Obviously $\operatorname{dom}\mathscr{R}$ may be empty for some γ, but for all γ sufficiently large, $\operatorname{dom}\mathscr{R} \neq \emptyset$. In the following, we will develop a framework for studying matrix inequalities of the form $\mathscr{R}(X) > 0$ and the corresponding matrix equations.

3. Positive and resolvent positive operators

Let $\mathscr{H}^n$ be endowed with the Frobenius inner product $\langle X, Y\rangle = \operatorname{trace}(XY)$ and the corresponding norm $\|X\| = \langle X, X\rangle^{1/2}$. By $\mathscr{H}^n_+ := \{X \in \mathscr{H}^n \mid X \geqslant 0\}$ we denote the closed convex cone of non-negative definite matrices and by $\operatorname{int}(\mathscr{H}^n_+)$ its interior, i.e. the open cone of positive definite matrices. The cone $\mathscr{H}^n_+$ induces the partial ordering on $\mathscr{H}^n : X \geqslant Y$, if $X - Y \in \mathscr{H}^n_+$. Moreover, the vector-space topology of $\mathscr{H}^n$ is generated by the open order intervals $]-\frac{1}{k}Y, \frac{1}{k}Y[= \{X \in \mathscr{H}^n \mid -\frac{1}{k}Y < X < \frac{1}{k}Y\}$, where Y is an arbitrary fixed positive definite matrix and $k \in \mathbb{N}$; i.e. a subset U in $\mathscr{H}^n$ is open if and only if for every $X \in U$ there exists $k \in \mathbb{N}$ such that $X +]-\frac{1}{k}Y, \frac{1}{k}Y[\subset U$.

Proposition 3.1.
(i) *The cone* $\mathscr{H}^n_+$ *is proper* (*compare* [3]), *which means that it is solid*, *i.e.* $\operatorname{int}(\mathscr{H}^n_+) \neq \emptyset$, *and pointed*, *i.e.* $\mathscr{H}^n_+ \cap -\mathscr{H}^n_+ = \{0\}$.
(ii) *The cone* $\mathscr{H}^m_+$ *is self-dual in* $\mathscr{H}^n_+$ (*compare* [2, 3]), *i.e.*

$$\mathscr{H}_+^n = \left(\mathscr{H}_+^n\right)^* := \{X \in \mathscr{H}^n \mid \forall Y \in \mathscr{H}_+^n : \langle X, Y\rangle \geqslant 0\}.$$

(iii) *If $X, Y \in \mathscr{H}_+^n$ and $\langle X, Y\rangle = 0$, then $XY = YX = 0$. If additionally $Y > 0$, then $X = 0$.*

Proof. Statement (i) is obvious. To prove (ii) and (iii), let $X, Y \in \mathscr{H}_+^n$ be arbitrary and choose $Z \in \mathbb{K}^{m\times n}$ such that $Y = Z^*Z$. Then $\langle X, Y\rangle = \operatorname{trace} ZXZ^*$ and $ZXZ^* \in \mathscr{H}_+^m$, whence the trace is non-negative, i.e. $\langle X, Y\rangle \geqslant 0$ and so $X \in (\mathscr{H}_+^n)^*$. If in this case $\operatorname{trace} ZXZ^* = 0$, then $ZXZ^* = 0$, since all eigenvalues necessarily vanish, and thus $ZX = XZ^* = 0$, which shows $XY = YX = 0$. The second statement of (iii) follows directly from (i).

If $X \notin \mathscr{H}_+^n$, then X has a negative eigenvalue $\lambda < 0$. Let $Z \in \mathbb{K}^n$ be a corresponding eigenvector and $Y := ZZ^* \in \mathscr{H}_+^n\backslash\{0\}$. Then $\langle X, Y\rangle = \operatorname{trace} Z^*XZ = \lambda\|Z\|^2 < 0$, i.e. $X \notin (\mathscr{H}_+^n)^*$. $\square$

For a linear operator $\mathscr{T}$ on a finite-dimensional vector space, let $\sigma(\mathscr{T})$ denote the spectrum, $\rho(\mathscr{T}) = \max\{|\lambda|; \lambda \in \sigma(\mathscr{T})\}$ the spectral radius, and $\beta(\mathscr{T}) = \max\{\mathrm{Re}(\lambda); \lambda \in \sigma(\mathscr{T})\}$ the spectral abscissa. The identity map is denoted by I, irrespective of the space it acts on.

Definition 3.2. A linear operator $\mathscr{T} : \mathscr{H}^n \to \mathscr{H}^m$ is called *positive* ($\mathscr{T} \geqslant 0$) if it maps $\mathscr{H}_+^n$ to $\mathscr{H}_+^m$, and *strictly positive* ($\mathscr{T} > 0$) if it maps $\mathrm{int}(\mathscr{H}_+^n)$ to $\mathrm{int}(\mathscr{H}_+^m)$. If $n = m$, then $\mathscr{T}$ is called *inverse positive* if $\mathscr{T}^{-1}$ exists and is (strictly) positive, and *resolvent positive* (compare [1]), if for all sufficiently large $\alpha > 0$ the operator $\alpha I - \mathscr{T}$ is inverse positive. For linear operators $\mathscr{S}, \mathscr{T} : \mathscr{H}^n \to \mathscr{H}^m$ we write $\mathscr{S} \geqslant \mathscr{T}$ if $\mathscr{S} - \mathscr{T}$ is positive.

Remark 3.3. Since $\mathscr{H}_+^n$ is self-dual, the adjoint operator $\mathscr{T}^*$ has the same positivity properties as $\mathscr{T}$.

Examples 3.4.

(i) Let $A_0 \in \mathbb{K}^{n\times n}$. Then the operator $\Pi : \mathscr{H}^n \to \mathscr{H}^n$ defined by $\Pi(X) := A_0^*XA_0$ is positive. If A_0 is non-singular, then it is also inverse positive. In particular, the zero map is positive and the identity map is both positive and inverse positive.

(ii) All positive operators Π are also resolvent positive, since for $\alpha > \rho(\Pi)$ the resolvent $(\alpha I - \Pi)^{-1} = \sum_{k=0}^{\infty} \alpha^{-(k+1)}\Pi^k$ is positive. This observation (together with Theorem 3.5) can be used to prove Theorem 3.6 (see [18]).

(iii) Important examples of resolvent positive operators which are not positive are provided by *Lyapunov operators*. Given $A \in \mathbb{K}^{n\times n}$ the associated Lyapunov operator $\mathscr{L}_A$ is defined by

$$\mathscr{L}_A : \mathscr{H}^n \to \mathscr{H}^n, \quad \mathscr{L}_A(X) := A^*X + XA. \tag{5}$$

Lyapunov operators play an important role in stability theory. It is well known (e.g. [13, Theorem 2.2.1]), that $\mathscr{L}_A$ is inverse negative, i.e. $-\mathscr{L}_A = \mathscr{L}_{-A}$ is inverse positive if and only if $\sigma(A) \subset \mathbb{C}_- = \{s \in \mathbb{C} \mid \operatorname{Re} s < 0\}$. In this case, the following representation of $\mathscr{L}_A^{-1}$ is well known (e.g. [13]):

$$\mathscr{L}_A^{-1}(Y) = -\int_0^\infty \mathrm{e}^{tA^*} Y \mathrm{e}^{tA}\, \mathrm{d}t. \tag{6}$$

Given any $A \in \mathbb{K}^{n\times n}$, the resolvent $(\alpha I - A)^{-1}$ is positive for $\alpha > 2\beta(A)$ because $\alpha I - \mathscr{L}_A = -\mathscr{L}_{A-\frac{\alpha}{2}I}$. It also follows from the definition that $\mathscr{L}_A(\mathscr{H}_+^n) \cap \operatorname{int} \mathscr{H}_+^n = \mathscr{L}_A(\operatorname{int} \mathscr{H}_+^n) \cap \operatorname{int} \mathscr{H}_+^n$. Furthermore, if $\sigma(A) \cap \overline{\mathbb{C}_-} \neq \emptyset$, then it is easily seen that $\mathscr{L}_A(\mathscr{H}_+^n) \cap \operatorname{int} \mathscr{H}_+^n = \emptyset$. For $\alpha \leqslant 2\beta(A)$ therefore $(\alpha I - \mathscr{L}_A)^{-1}(\operatorname{int} \mathscr{H}_+^n) \cap \mathscr{H}_+^n = \emptyset$. This can also be derived from Theorem 3.6 below (compare [18]).

We now cite two results, that hold in arbitrary finite-dimensional spaces ordered by proper cones. For convenience of application we specialize them to the present context. The first result is the well-known Theorem of Krein and Rutman, which generalizes the Perron–Frobenius Theorem (see e.g. [20]).

Theorem 3.5. *Let $\mathscr{T} : \mathscr{H}^n \to \mathscr{H}^n$ be a positive linear operator. Then $\rho(\mathscr{T}) \in \sigma(\mathscr{T})$ and there exists a matrix $V \in \mathscr{H}_+^n \backslash \{0\}$ such that $\mathscr{T}(V) = \rho(\mathscr{T})V$. If further $\mathscr{S}$ is a linear operator with $\mathscr{S} \geqslant \mathscr{T}$, then $\mathscr{S}$ is positive, and $\rho(\mathscr{S}) \geqslant \rho(\mathscr{T})$.*

The next result can be derived from the previous one and is due to Schneider [18].

Theorem 3.6. *Let $\mathscr{S}, \mathscr{T} : \mathscr{H}^n \to \mathscr{H}^n$ be linear transformations such that $\mathscr{S}$ is positive and either $\mathscr{T}$ is inverse positive or $\mathscr{T}(\operatorname{int} \mathscr{H}_+^n) \cap \operatorname{int} \mathscr{H}_+^n = \emptyset$. Then the following are equivalent:*
- (i) *$\mathscr{T}$ is inverse positive, and $\rho(\mathscr{T}^{-1}\mathscr{S}) < 1$.*
- (ii) *$\mathscr{T} - \mathscr{S}$ is inverse positive.*
- (iii) *$(\mathscr{T} - \mathscr{S})(\operatorname{int} \mathscr{H}_+^n) \cap \operatorname{int} \mathscr{H}_+^n \neq \emptyset$.*

We use these two results to prove similar statements for resolvent positive operators.

Theorem 3.7. *Let $\mathscr{T} : \mathscr{H}^n \to \mathscr{H}^n$ be a resolvent positive linear operator. Then the following assertions hold:*
- (i) *$\beta(\mathscr{T}) \in \sigma(\mathscr{T})$ and there exists a matrix $V \in \mathscr{H}_+^n \backslash \{0\}$ such that $\mathscr{T}(V) = \beta(\mathscr{T})V$.*
- (ii) *If $\mathscr{S} : \mathscr{H}^n \to \mathscr{H}^n$ is a linear operator with $\mathscr{S} \geqslant \mathscr{T}$, then $\mathscr{S}$ is resolvent-positive and $\beta(\mathscr{S}) \geqslant \beta(\mathscr{T})$.*
- (iii) *$\alpha I - \mathscr{T}$ is inverse positive $\iff \alpha > \beta(\mathscr{T}) \iff \sigma(\mathscr{T} - \alpha I) \subset \mathbb{C}_-$.*

(iv) *If* $\alpha > \beta(\mathscr{T})$, *then*

$$\beta(\mathscr{T}) = \alpha - \frac{1}{\rho((\alpha I - \mathscr{T})^{-1})}.$$

Proof. (i) For sufficiently large $\alpha > 0$

$$\alpha^2(\alpha I - \mathscr{T})^{-1} = \alpha \sum_{k=0}^{\infty} \frac{1}{\alpha^k}\mathscr{T}^k = \alpha I + \mathscr{T} + \Phi(\alpha) \quad (\text{where } \Phi(\alpha) \overset{\alpha\to\infty}{\longrightarrow} 0)$$

is by assumption a positive operator. Thus,

$$\rho_{\alpha,\mathscr{T}} := \rho(\alpha^2(\alpha I - \mathscr{T})^{-1}) \in \sigma(\alpha^2(\alpha I - \mathscr{T})^{-1})$$

and $\rho_{\alpha,\mathscr{T}} = \alpha + \lambda_\alpha$ for some $\lambda_\alpha \in \sigma(\mathscr{T} + \Phi(\alpha))$. Since $\rho_{\alpha,\mathscr{T}}$ is real and has maximal real part in $\sigma(\alpha^2(\alpha I - \mathscr{T})^{-1})$, it follows that also λ_α is real and has maximal real part in $\sigma(\mathscr{T} + \Phi(\alpha))$, i.e. $\lambda_\alpha = \beta_{\alpha,\mathscr{T}} = \beta(\mathscr{T} + \Phi(\alpha))$. Moreover, there exists, by Theorem 3.5, a non-negative unit eigenvector V_α such that $\alpha^2(\alpha I - \mathscr{T})^{-1}(V_\alpha) = (\alpha + \beta_{\alpha,\mathscr{T}})V_\alpha$. Hence $(\mathscr{T} + \Phi(\alpha))(V_\alpha) = \beta_{\alpha,\mathscr{T}} V_\alpha$. If α tends to infinity, then $\sigma(\mathscr{T} + \Phi(\alpha))$ tends to $\sigma(\mathscr{T})$ and thus $\beta_{\alpha,\mathscr{T}}$ converges to $\beta(\mathscr{T})$. Choosing an appropriate sequence $\alpha_k \to \infty$, we can assume that V_{α_k} converges to some $V \geqslant 0$, which by continuity is an eigenvector of $\mathscr{T}$ corresponding to $\beta(\mathscr{T})$.

(ii) Suppose $\mathscr{S} \geqslant \mathscr{T}$. To prove that $\mathscr{S}$ is resolvent positive, choose α large enough such that $\alpha I - \mathscr{T}$ is inverse positive and $\rho((\alpha I - \mathscr{T})^{-1}(\mathscr{S} - \mathscr{T}) < 1$. Applying Theorem 3.6 with $\mathscr{S} := I$ (inverse positive) and $\mathscr{T} := (\alpha I - \mathscr{T})^{-1}(\mathscr{S} - \mathscr{T})$ (positive), we obtain that $(I - (\alpha I - \mathscr{T})^{-1}(\mathscr{S} - \mathscr{T}))$ is inverse positive. Therefore, the product

$$(\alpha I - \mathscr{S})^{-1} = (I - (\alpha I - \mathscr{T})^{-1}(\mathscr{S} - \mathscr{T}))^{-1}(\alpha I - \mathscr{T})^{-1}$$

is positive, i.e. $\mathscr{S}$ is resolvent positive. Now the monotonicity statement follows from:

$$(\alpha I - \mathscr{S})^{-1} - (\alpha I - \mathscr{T})^{-1} = (\alpha I - \mathscr{S})^{-1}(\mathscr{S} - \mathscr{T})(\alpha I - \mathscr{T})^{-1} \geqslant 0,$$

because this implies $\rho_{\alpha,\mathscr{S}} \geqslant \rho_{\alpha,\mathscr{T}}$ and thus $\beta_{\alpha,\mathscr{S}} \geqslant \beta_{\alpha,\mathscr{T}}$ for sufficiently large α.

(iii) The second equivalence follows directly from the definition of the spectral abscissa. We now prove the first equivalence. Assume that $\alpha I - \mathscr{T}$ is inverse positive, but $\alpha \leqslant \beta(\mathscr{T})$. Then $\alpha I - \mathscr{T}$ must be regular and hence $\alpha < \beta(\mathscr{T})$ by (i). Applying again (i), let $V \in \mathscr{H}^n_+\backslash\{0\}$ be a non-negative eigenvector so that $\mathscr{T}(V) = \beta(\mathscr{T})V$. Then $(\alpha I - \mathscr{T})V = (\alpha - \beta)V \leqslant 0$, whence $(\alpha I - \mathscr{T})^{-1}$ is not positive. This proves the implication '$\Rightarrow$'.

'$\Leftarrow$': Since $\mathscr{T}$ is resolvent positive, there exists an $\alpha_0 \in \mathbb{R}$ such that $\alpha I - \mathscr{T}$ is inverse positive for all $\alpha > \alpha_0$. We assume α_0 to be chosen minimal with this property and want to show that it coincides with $\beta(\mathscr{T})$. From the implication '$\Rightarrow$' it is obvious, that $\alpha_0 \geqslant \beta(\mathscr{T})$. We assume $\alpha_0 > \beta(\mathscr{T})$. Then $(\alpha_0 I - \mathscr{T})^{-1}$ exists and is positive by continuity. Now we choose $\tilde{\alpha} \in [\beta(\mathscr{T}), \alpha_0[$ such that $\alpha_0 - \tilde{\alpha} < 1/\rho((\alpha_0 I - \mathscr{T})^{-1})$. For this $\tilde{\alpha}$ we have

$$(\tilde{\alpha}I - \mathscr{T})^{-1} = (\alpha_0 I - \mathscr{T})^{-1}(I - (\alpha_0 - \tilde{\alpha})(\alpha_0 I - \mathscr{T})^{-1})^{-1},$$

which again by Theorem 3.6 is the product of two positive operators and thus positive — in contradiction to the minimality of α_0.

(iv) Assume $\alpha > \beta(\mathscr{T})$. Then $(\alpha I - \mathscr{T})^{-1} \geqslant 0$ by (iii) and so $\rho := \rho((\alpha I - \mathscr{T})^{-1}) \in \sigma((\alpha I - \mathscr{T})^{-1})$ by Theorem 3.5. It follows that $\alpha - 1/\rho \in \sigma(\mathscr{T})$, and hence $\alpha - 1/\rho \leqslant \beta(\mathscr{T})$, i.e. $(\alpha - \beta(\mathscr{T}))^{-1} \geqslant \rho$. But by (i) we have $\beta(\mathscr{T}) \in \sigma(\mathscr{T})$ and so $(\alpha - \beta(\mathscr{T}))^{-1} \in \sigma((\alpha I - \mathscr{T})^{-1})$ which implies $(\alpha - \beta(\mathscr{T}))^{-1} \leqslant \rho$. This proves (iv). □

Corollary 3.8. *Let $\mathscr{L} : \mathscr{H}^n \to \mathscr{H}^n$ be resolvent positive and $\Pi : \mathscr{H}^n \to \mathscr{H}^n$ be positive. Then the following are equivalent*:

(i) *$\mathscr{L} + \Pi$ is stable, i.e. $\sigma(\mathscr{L} + \Pi) \subset \mathbb{C}_-$.*
(ii) *$-(\mathscr{L} + \Pi)$ is inverse positive.*
(iii) $\exists X > 0 : (\mathscr{L} + \Pi)(X) < 0$.
(iv) *$\sigma(\mathscr{L}) \subset \mathbb{C}_-$ and $\rho(\mathscr{L}^{-1}\Pi) < 1$.*
(v) *$\sigma(\mathscr{L}) \subset \mathbb{C}_-$ and* $\det(\mathscr{L} + \tau\Pi) \neq 0$ *for $\tau \in [0, 1]$.*

Proof. (i) $\Leftrightarrow$ (ii): By Theorem 3.7(ii) $\mathscr{L} + \Pi$ is resolvent positive and thus by Theorem 3.7(iii) the first two statements are equivalent.

(ii) $\Leftrightarrow$ (iii): Obviously (ii) implies (iii). If (ii) fails, then (i) fails, and by Theorem 3.7(i) there exist a matrix $0 \neq V \geqslant 0$ and a $\lambda \geqslant 0$ such that $(\mathscr{L} + \Pi)^*(V) = \lambda V$. Now assume that (iii) holds, i.e. $\exists X > 0 : (\mathscr{L} + \Pi)(X) < 0$. Then $0 \geqslant \langle V, (\mathscr{L} + \Pi)(X)\rangle = \langle \lambda V, X\rangle \geqslant 0$, whence $V = 0$ by Proposition 3.1, which is a contradiction. Thus (iii) implies (ii).

(ii) $\Leftrightarrow$ (iv): Applying Theorem 3.6 with $\mathscr{T} = -\mathscr{L}$, $\mathscr{S} = \Pi \geqslant 0$ and observing that by Theorem 3.7(iii) $-\mathscr{L}$ is inverse positive if and only if $\sigma(\mathscr{L}) \subset \mathbb{C}_-$ we see that assertions (ii) and (iv) are equivalent.

(iv) $\Leftrightarrow$ (v): (iv) $\Rightarrow$ (v) follows, since $\rho(\mathscr{L}^{-1}\Pi) < 1 \leqslant \tau^{-1}$ implies $\det(I + \mathscr{L}^{-1}\tau\Pi) \neq 0$. To prove (v) $\Rightarrow$ (iv) assume (v) and let $\rho := \rho(\mathscr{L}^{-1}\Pi) \geqslant 1$. As $-\mathscr{L}^{-1}$ is positive by Theorem 3.7(iii), $-\mathscr{L}^{-1}\Pi$ is positive and we have by Theorem 3.5 $\rho \in \sigma(-\mathscr{L}^{-1}\Pi)$, i.e. $\det(\mathscr{L}_A^{-1}\Pi + \rho I) = 0$. Thus, the determinant condition in (iv) fails for $\tau = \rho^{-1} \in [0, 1]$. □

Remark 3.9. In a special case the equivalence of (i), (ii), and (iii) in Corollary 3.8 follows from a result in stochastic analysis given by Khasminskij [14]. If Π is of the special form

$$\Pi(X) = \sum_{i=1}^{N} A_0^{(i)*} X A_0^{(i)}, \quad A_0^{(i)} \in \mathbb{K}^{n\times n}, \tag{7}$$

then each of the conditions (i), (ii), and (iii) is equivalent to the exponential mean-square stability (see Definition 2.1) of the linear Itô differential equation

$$\mathrm{d}x(t) = Ax(t)\,\mathrm{d}t + \sum_{i=1}^{N} A_0^{(i)} x(t)\,\mathrm{d}w_i(t).$$

Operators of the form (7) are called *completely positive* operators. They constitute a proper subcone of the cone of positive operators (cf. [4]).

Corollary 3.8 thus can be regarded as a generalization of Lyapunov's stability theorem for deterministic differential equations (if $\Pi = 0$). As for Lyapunov equations, we can weaken the definiteness conditions in Corollary 3.8(iii), if (A, G) is observable.

Proposition 3.10. *Let* $(A, G) \in \mathbb{K}^{n\times n} \times \mathscr{H}_+^n$ *be observable, i.e.* $\bigcap_{i=1}^n \operatorname{Ker} GA^{i-1} = \{0\}$, *and assume*

$$\exists X \leqslant 0 : \mathscr{L}_A(X) + \Pi(X) \geqslant G. \tag{8}$$

Then $X < 0$ *and* $\mathscr{L}_A + \Pi$ *is stable.*

Proof. Let $X \leqslant 0$ satisfy (8). Then

$$A^*X + XA \geqslant G. \tag{9}$$

It is well known (e.g. [13, Theorem 2.4.7]) that this implies $X < 0$ and $\sigma(A) \subset \mathbb{C}_-$.

To complete the proof, it suffices by Corollary 3.8 to show that $\det(\mathscr{L}_A + \tau\Pi) \neq 0$ for $\tau \in [0, 1]$. Suppose $\mathscr{L}_A(X_0) + \tau\Pi(X_0) = 0$ and consider the convex combination $X_\alpha := \alpha X + (1-\alpha)X_0$. By our assumptions

$$\mathscr{L}_A(X_\alpha) \geqslant \alpha G - (1-\alpha)\tau\Pi(X). \tag{10}$$

We want to show $X_0 \leqslant 0$ and $X_0 \geqslant 0$. Assume first $X_0 \nleqslant 0$. Since $X_1 = X < 0$, there exists $\alpha_0 = \max\{\alpha \in]0,1] \mid X_\alpha \not< 0\}$. Then $X_{\alpha_0} \leqslant 0$ and $\mathscr{L}_A(X_{\alpha_0}) \geqslant \alpha_0 G$, whence by a well-known argument $X_{\alpha_0} < 0$ — a contradiction.

Now assume $X_0 \ngeqslant 0$, then the last argument can be repeated with X replaced by $-X > 0$ and inverted order. Thus $\operatorname{Ker}(\mathscr{L}_A + \tau\Pi) = \{0\}$. □

4. Concave maps

In this section, we define concave operators on the ordered vector space $\mathscr{H}^n$ and prove some simple properties needed later. The concept is the same as in the scalar case and by a Hahn–Banach type argument we can always fall back upon this case.

Definition 4.1. Let $\mathscr{K}$ be a convex subset of $\mathscr{H}^n$. A mapping $\mathscr{T} : \mathscr{K} \to \mathscr{H}^m$ is said to be *concave* if

$$\begin{aligned} &\forall X_0, X_1 \in \mathscr{K},\quad t \in]0,1[: \\ &\quad t\mathscr{T}(X_1) + (1-t)\mathscr{T}(X_0) \leqslant \mathscr{T}(tX_1 + (1-t)X_0) \end{aligned} \tag{11}$$

and *strictly concave* if inequality (11) is strict.

The following theorem is an immediate consequence of Proposition 3.1 and corresponding results for real-valued maps.

Theorem 4.2. *Let $\mathscr{K}$ be a convex subset of $\mathscr{H}^n$.*
(i) *A map $\mathscr{T}: \mathscr{K} \to \mathscr{H}^m$ is (strictly) concave if and only if for all non-zero $V \in \mathscr{H}^m_+$ the real-valued function $f: \mathscr{K} \to \mathbb{R}$ defined by $H \mapsto \langle V, \mathscr{T}(H)\rangle$ is (strictly) concave.*
(ii) *If $\mathscr{T}: \mathscr{U} \to \mathscr{H}^m$ is Fréchet differentiable on an open neighbourhood $\mathscr{U}$ of $\mathscr{K}$, then $\mathscr{T}$ is concave on $\mathscr{K}$ if and only if*

$$\forall X, Y \in \mathscr{K}: \mathscr{T}(Y) - \mathscr{T}(X) \leqslant \mathscr{T}'_X(Y - X), \tag{12}$$

where $\mathscr{T}'_X: \mathscr{H}^n \to \mathscr{H}^n$ denotes the Fréchet derivative of $\mathscr{T}$ at the point $X \in \mathscr{K}$. $\mathscr{T}$ is strictly concave if and only if strict inequality holds in (12).

Proof. (i) Inequality (11) holds if and only if for all $X_0, X_1 \in \mathscr{K}$, $t \in]0,1[$ and all $V \geqslant 0,\ V \neq 0$:

$$\underbrace{\langle V, t\mathscr{T}(X_1) + (1-t)\mathscr{T}(X_0)\rangle}_{=tf(X_1)+(1-t)f(X_0)} \leqslant \underbrace{\langle V, \mathscr{T}(tX_1 + (1-t)X_0)\rangle}_{=f(tX_1+(1-t)X_0)}; \tag{13}$$

with strict inequality if and only if (11) is strict (by Proposition 3.1).

The criterion (ii) is well known in the scalar-valued case: Under the differentiability assumption in (ii) the concavity of $f(\cdot) := \langle V, T(\cdot)\rangle$ is equivalent to condition (12) with $\mathscr{T}$ replaced by f. Since $f'_X(\cdot) = \langle V, T'_X(\cdot)\rangle$, it follows from (i) that $\mathscr{T}$ is concave if and only if $\langle V, \mathscr{T}(Y) - \mathscr{T}(X)\rangle \geqslant \langle V, \mathscr{T}'_X(Y - X)\rangle$ for all $V \geqslant 0$ and $X, Y \in \mathscr{K}$. By Proposition 3.1 the latter condition is equivalent to (12). The strict case can be obtained analogously. □

If $\mathscr{T}$ is concave, but not strictly concave, then for some positive functional $0 \neq V \in \mathscr{H}^n_+$ the graph of the scalar-valued function $\langle V, \mathscr{T}(\cdot)\rangle$ contains a straight line. We will now show that, in all points of this straight line, the graph has the same tangential hyperplane.

Lemma 4.3. *Let $\mathscr{K}$ be a convex subset of $\mathscr{H}^n$ and assume $\mathscr{T}: \mathscr{U} \to \mathscr{H}^m$ to be Fréchet differentiable on an open neighbourhood $\mathscr{U}$ of $\mathscr{K}$ and concave on $\mathscr{K}$. If for some $V \in \mathscr{H}^m_+$ and $X, Y \in \operatorname{int}\mathscr{K}$:*

$$\langle V, \mathscr{T}(Y) - \mathscr{T}(X)\rangle = \langle V, \mathscr{T}'_X(Y - X)\rangle, \tag{14}$$

then $\langle V, \mathscr{T}'_X(\cdot)\rangle = \langle V, \mathscr{T}'_Y(\cdot)\rangle$, i.e. $(\mathscr{T}'_X)^*(V) = (\mathscr{T}'_Y)^*(V)$.

Proof. Set $f(X) := \langle V, \mathscr{T}(X)\rangle$. Then by (14) $f(Y) - f(X) = f'_X(Y - X)$ and by (12)

$$
\begin{aligned}
\forall Z \in \mathscr{K}: \quad f(Z) &\leqslant f(X) + f_X'(Z - X) \\
&= f(X) + f_X'(Y - X) + f_X'(Z - Y) \\
&= f(Y) + f_X'(Z - Y).
\end{aligned}
\tag{15}
$$

Now let $H \in \mathscr{H}^n$ such that $Y \pm H \in \mathscr{K}$. We write $\phi(\cdot)$ for the remainder term of the first-order Taylor expansion of f at Y. Then we have by (15)

$$
f(Y \pm tH) = f(Y) \pm t f_Y'(H) + \phi(\pm tH) \leqslant f(Y) \pm t f_X'(H), \quad t \in]0, 1],
$$

whence $\pm(f_Y' - f_X')(H) + \frac{1}{t}\phi(\pm tH) \leqslant 0$, i.e. $\pm(f_Y' - f_X')(H) \leqslant 0$ since $\frac{1}{t}\phi(\pm tH) \xrightarrow{t \to 0} 0$. Thus $f_Y'(H) = f_X'(H)$. $\square$

5. Derivatives of Riccati type rational matrix operators

In this section, we introduce a class of rational matrix operators which comprises all operators of form (3) and derive some properties of their derivatives which will be important in the sequel. Let P, S, and Q be affine linear matrix operators from $\mathscr{H}^n$ to $\mathscr{H}^n$, $\mathbb{K}^{n\times\ell}$, and $\mathscr{H}^\ell$, respectively, of the following form:

$$
\begin{aligned}
&P(X) = A^*X + XA + \Pi_1(X) + P_0, \\
&S(X) = XB + \Sigma(X) + S_0, \\
&Q(X) = \Pi_2(X) + Q_0,
\end{aligned}
\tag{16}
$$

where $A \in \mathbb{K}^{n\times n}$, $P_0 \in \mathscr{H}^n$, B, $S_0 \in \mathbb{K}^{n\times\ell}$, and $Q_0 \in \mathscr{H}^\ell$. *We assume that the linear operators Π_1, Π_2, and Σ together form a positive linear operator*

$$
\Pi : \mathscr{H}^n \to \mathscr{H}^{n+\ell}, \quad \Pi(X) = \begin{bmatrix} \Pi_1(X) & \Sigma(X) \\ \Sigma(X)^* & \Pi_2(X) \end{bmatrix}. \tag{17}
$$

In particular, $\Pi_1 : \mathscr{H}^n \to \mathscr{H}^n$ and $\Pi_2 : \mathscr{H}^n \to \mathscr{H}^\ell$ must be positive. We define

$$
\begin{aligned}
&M := \begin{bmatrix} P_0 & S_0 \\ S_0^* & Q_0 \end{bmatrix} \in \mathscr{H}^{n+\ell}; \quad \Lambda : \mathscr{H}^n \to \mathscr{H}^{n+\ell}, \\
&\Lambda(X) := \begin{bmatrix} A^*X + XA & XB \\ B^*X & 0 \end{bmatrix}, \quad X \in \mathscr{H}^n.
\end{aligned}
\tag{18}
$$

For applications to stochastic control it is important that we do not impose any restrictions on the inertia of M or any of its submatrices.

Our object is to study the rational matrix operator $\mathscr{R} : \operatorname{dom} \mathscr{R} \to \mathscr{H}^n$, given by

$$
\begin{aligned}
&\mathscr{R}(X) = P(X) - S(X)Q(X)^{-1}S(X)^*, \\
&X \in \operatorname{dom}\mathscr{R} = \{X \in \mathscr{H}^n \mid Q(X) > 0\}.
\end{aligned}
\tag{19}
$$

We assume $\operatorname{dom}\mathscr{R} \neq \emptyset$. Note that $X \in \operatorname{dom}\mathscr{R}$ implies $Y \in \operatorname{dom}\mathscr{R}$ for all $Y \geqslant X$ and that $\operatorname{dom}\mathscr{R}$ is open and convex.

We are interested in the rational matrix equation

$$\mathscr{R}(X) = 0, \tag{20}$$

and the corresponding strict and non-strict inequalities. The definite version of Eq. (20) with $\Sigma = 0$ and $M > 0$ was studied first by Wonham in [24], see also [6,10].

Remark 5.1.

(i) $\mathscr{R}(X)$ is the Schur complement of the matrix

$$R(X) = \Lambda(X) + \Pi(X) + M = \begin{bmatrix} P(X) & S(X) \\ S(X)^* & Q(X) \end{bmatrix}.$$

Hence $X \in \operatorname{dom} \mathscr{R}$ solves the non-linear matrix inequality $\mathscr{R}(X) \geqslant 0$ (resp. $\mathscr{R}(X) > 0$) if and only if it solves the higher dimensional linear matrix inequality $R(X) \geqslant 0$ (resp. $R(X) > 0$).

(ii) In the case of the stochastic bounded real lemma considered in Section 2, we have

$$\Pi(X) = \begin{bmatrix} \sum_{i=1}^N A_0^{i\,*} X A_0^i & \sum_{i=1}^N A_0^{i\,*} X B_0^i \\ \sum_{i=1}^N B_0^{i\,*} X A_0^i & \sum_{i=1}^N B_0^{i\,*} X B_0^i \end{bmatrix} = \sum_{i=1}^N \begin{bmatrix} A_0^{i\,*} \\ B_0^{i\,*} \end{bmatrix} X \begin{bmatrix} A_0^i & B_0^i \end{bmatrix},$$

$$M = \begin{bmatrix} -C^*C & -C^*D \\ -D^*C & \gamma^2 I - D^*D \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 I \end{bmatrix} - \begin{bmatrix} C^* \\ D^* \end{bmatrix} [C \quad D]. \tag{21}$$

The first equation shows that our positivity assumption is satisfied in this case.

(iii) Eq. (20) reduces to the continuous algebraic Riccati equation (CARE)

$$A^*X + XA + P_0 - (XB + S_0)Q_0^{-1}(XB + S_0)^* = 0 \tag{22}$$

if $\Pi = 0$, and to the discrete algebraic Riccati equation (DARE)

$$\begin{aligned} &{A_0}^* X A_0 - X + P_0 - \left({A_0}^* X B_0 + S_0\right)\left({B_0}^* X B_0 + Q_0\right)^{-1} \\ &\quad \times \left({A_0}^* X B_0 + S_0\right)^* = 0 \end{aligned} \tag{23}$$

if $A = -\frac{1}{2}I$, $B = 0$, and $\Pi_1(X) = {A_0}^* X A_0$, $\Pi_2(X) = {B_0}^* X B_0$, and $\Sigma(X) = {A_0}^* X B_0$.

Neither the algebraic Riccati equations (22) and (23) nor Wonham's equation need to be solvable. But, if they are, and certain stabilizability conditions are fulfilled, then they can be solved by a Newton–Kantorovich iteration starting with an arbitrary stabilizing matrix X_0 (compare [15]). We will show in Section 6 that the same is true for the general equation (20). To this end, we need to calculate the Fréchet derivative $\mathscr{R}'_X$ of $\mathscr{R}$ at points $X \in \operatorname{dom} \mathscr{R}$. In order to condense the notation, we write P_X, S_X, and Q_X instead of $P(X)$, $S(X)$, and $Q(X)$. Moreover, we write

$$\Pi_X(H) := \begin{bmatrix} I \\ -Q_X^{-1} S_X^* \end{bmatrix}^* \Pi(H) \begin{bmatrix} I \\ -Q_X^{-1} S_X^* \end{bmatrix} \text{ and } A_X := A - BQ_X^{-1} S_X^* \tag{24}$$

for $X \in \operatorname{dom} \mathscr{R}$.

Proposition 5.2. *For $X \in \operatorname{dom} \mathscr{R}$ the derivative of $\mathscr{R}$ at X is a resolvent positive operator on $\mathscr{H}^n$ given by $\mathscr{R}'_X = \mathscr{L}_{A_X} + \Pi_X$.*

Proof. By a direct calculation we have

$$\begin{aligned}\mathscr{R}'_X(H) =& P'_X(H) - S'_X(H) Q_X^{-1} S_X^* \\ &+ S_X Q_X^{-1} Q'_X(H) Q_X^{-1} S_X^* - S_X Q_X^{-1} S'_X(H)^* \\ =& A^*H + HA + \Pi_1(H) - HBQ_X^{-1}S_X^* - \Sigma(H)Q_X^{-1}S_X^* \\ &- S_X Q_X^{-1} B^* H - S_X Q_X^{-1} \Sigma(H)^* + S_X Q_X^{-1} \Pi_2(H) Q_X^{-1} S_X^* \\ =& \mathscr{L}_{A_X}(H) + \Pi_X(H)\end{aligned}$$

with Π_X and A_X from (24). Since $\mathscr{L}_{A_X}$ is resolvent positive by Example 3.4(iii) and Π_X is positive, the sum $\mathscr{L}_{A_X} + \Pi_X$ is resolvent positive by Theorem 3.7(ii). □

We introduce an appropriate stabilizability concept for the matrix operator $\mathscr{R}$.

Definition 5.3. A matrix $X \in \operatorname{dom} \mathscr{R}$ is called *stabilizing for* $\mathscr{R}$ if $\sigma(\mathscr{R}'_X) \subset \mathbb{C}_-$ and *almost stabilizing* if $\sigma(\mathscr{R}'_X) \subset \mathbb{C}_- \cup \mathrm{i}\mathbb{R}$. We call $\mathscr{R}$ *(almost) stabilizable* if there exists an (almost) stabilizing matrix X for $\mathscr{R}$.

Remark 5.4. (i) By Corollary 3.8 a matrix X is stabilizing if and only if A_X is stable and $\rho := \rho(\mathscr{L}_{A_X}^{-1} \Pi_X) < 1$. Suppose that A_X is stable so that $-\mathscr{L}_{A_X}^{-1}$ is a positive operator. By Theorem 3.5 there exists an eigenvector $P \geqslant 0$ such that $-\mathscr{L}_{A_X}^{-1} \Pi_X(P) = \rho P$. Thus $\rho < 1$ if and only if $\|\mathscr{L}_{A_X}^{-1} \Pi_X(P)\| < \|P\|$ holds for all eigenvectors $P \in \mathscr{H}^n$ of $-\mathscr{L}_{A_X}^{-1} \Pi_X$. In [24], Wonham has given the following sufficient condition for the solvability of the classical linear quadratic optimal control problem in the case of multiplicative noise:

$$\left\| \mathscr{L}_{A_X}^{-1}(\Pi_X(I)) \right\| = \left\| \int_0^\infty \mathrm{e}^{tA_X^*} \Pi_X(I) \mathrm{e}^{tA_X}\, \mathrm{d}t \right\| < 1. \tag{25}$$

This condition roughly means that the noise effect on the feedback system defined by X should not be too large. It implies that for all $P \in \mathscr{H}_+^n$ (not only for the eigenvectors of $-\mathscr{L}_{A_X}^{-1} \Pi_X$)

$$\left\| \mathscr{L}_{A_X}^{-1} \Pi_X(P) \right\| \leqslant \left\| \mathscr{L}_{A_X}^{-1} \Pi_X(\|P\| I) \right\| < \|P\|.$$

In [11] examples can be found illustrating that inequality (25) is not a necessary solvability condition.

(ii) In the special case that $\mathscr{R}(X) = 0$ is the CARE (22), a matrix X is stabilizing in the sense of Definition 5.3, if and only if all eigenvalues of $A_X = A - BQ_0^{-1}(S_0^* + B^*X)$ lie in the open left half plane. In fact, this follows from (i) since the condition $\rho := \rho(\mathscr{L}_{A_X}^{-1} \Pi_X) < 1$ is automatically satisfied because of $\Pi_X = 0$. Hence in this case our stabilizability concept coincides with the classical one, see e.g. [15,16].

(iii) The same holds true for the special case where $\mathscr{R}(X)=0$ is the DARE (23). In this case a matrix X is stabilizing, if and only if

$$\sigma\left(A_0-B_0\left(Q_0+B_0^*XB_0\right)^{-1}\left(S_0^*+B_0^*XA_0\right)\right)\subset\mathbb{D}:=\{s\in\mathbb{C}\mid |s|<1\}. \tag{26}$$

In fact, then $A_X=-(1/2)I$ and so $-\mathscr{L}_{A_X}=-\mathscr{L}_{-(1/2)I}$ is the identity, whereas

$$\Pi_X(H)=L^*HL,$$

where

$$L=[A_0B_0]\begin{bmatrix} I \\ -\left(Q_0+B_0^*XB_0\right)^{-1}\left(S_0^*+B_0^*XA_0\right)\end{bmatrix}.$$

So A_X is stable and $\rho(\mathscr{L}_{A_X}^{-1}\Pi_X)=\rho(\Pi_X)<1$ if and only if the map $H\mapsto L^*HL$ has spectral radius <1. But it is well known that the latter holds if and only if $\sigma(L)\subset\mathbb{D}$, i.e. (26) is satisfied.

We will now investigate the error of the first-order Taylor expansion of $\mathscr{R}$ and show that $\mathscr{R}$ is a concave operator on dom $\mathscr{R}$. It is convenient to write

$$M_Z:=\begin{bmatrix} I \\ -Q_Z^{-1}S_Z^*\end{bmatrix}^* M\begin{bmatrix} I \\ -Q_Z^{-1}S_Z^*\end{bmatrix} \tag{27}$$

for $Z\in\operatorname{dom}\mathscr{R}$, and

$$\begin{aligned}\Phi_Z(Y)&:=\begin{bmatrix} I \\ -Q_Z^{-1}S_Z^*\end{bmatrix}^*\begin{bmatrix} S_YQ_Y^{-1}S_Y^* & S_Y \\ S_Y^* & Q_Y\end{bmatrix}\begin{bmatrix} I \\ -Q_Z^{-1}S_Z^*\end{bmatrix}\\ &=\left(S_YQ_Y^{-1}-S_ZQ_Z^{-1}\right)Q_Y\left(Q_Y^{-1}S_Y^*-Q_Z^{-1}S_Z^*\right)\end{aligned} \tag{28}$$

for $Y,Z\in\operatorname{dom}\mathscr{R}$. Note that $\Phi_Z(Y)\geqslant 0$ for all $Y,Z\in\operatorname{dom}\mathscr{R}$, and $\Phi_Z(Z)=0$.

It turns out that $-\Phi_Z(Y)$ is the remainder term of the first-order Taylor expansion of $\mathscr{R}$ at Z in direction $Y-Z$, see Fig. 1 and assertion (iv) in the next proposition.

In particular the positivity of $\Phi_Z(Y)$ implies the concavity of $\mathscr{R}$. The remaining identities in the following proposition have only technical significance and are listed here for later use. Equalities (i), (iv) and (v) are illustrated in Fig. 1.

Proposition 5.5. *Let $Y,Z\in$ dom $\mathscr{R}$. Then the following identities hold*:

(i) $\mathscr{R}(Z)=\mathscr{R}'_Z(Z)+M_Z$.
(ii) $(\mathscr{R}'_Y-\mathscr{R}'_Z)(H)=M_Z-M_Y-\Phi_Z(H)+\Phi_Y(H)$.
(iii) $(\mathscr{R}'_Y-\mathscr{R}'_Z)(Y)=M_Z-M_Y-\Phi_Z(Y)$.
(iv) $\mathscr{R}(Y)=\mathscr{R}'_Z(Y)+M_Z-\Phi_Z(Y)$.
(v) $\mathscr{R}(Z)+R'_Z(Y-Z)=\mathscr{R}(Y)+\Phi_Z(Y)\geqslant\mathscr{R}(Y)$,
i.e. $\mathscr{R}$ is a concave map on dom $\mathscr{R}$.

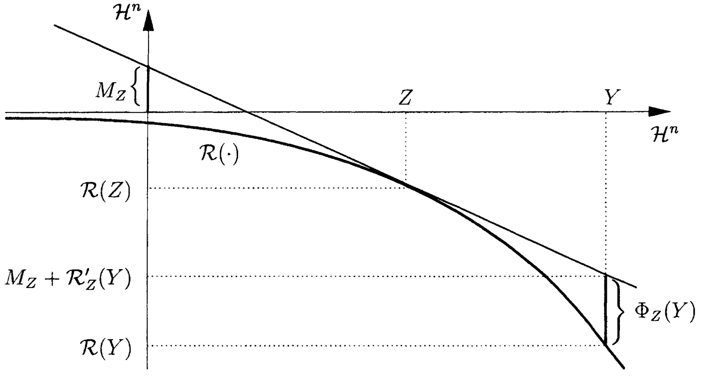

Fig. 1. Graph of $\mathscr{R}$ and its tangent at $Z \in \mathscr{H}^n$.

Proof. (i) This follows from a short calculation:

$$\begin{aligned}
\mathscr{R}(Z) &= P_Z - S_Z Q_Z^{-1} S_Z^* - S_Z Q_Z^{-1} S_Z^* + S_Z Q_Z^{-1} Q_Z Q_Z^{-1} S_Z^* \\
&= A^* Z + ZA + \Pi_1(Z) P_0 \\
&\quad - ZBQ_Z^{-1} S_Z^* - \Sigma(Z) Q_Z^{-1} S_Z^* - S_0 Q_Z^{-1} S_Z^* \\
&\quad - S_Z Q_Z^{-1} B^* Z - S_Z Q_Z^{-1} \Sigma(Z)^* - S_Z Q_Z^{-1} S_0^* \\
&\quad + S_Z Q_Z^{-1} \Pi_2(Z) Q_Z^{-1} S_Z^* + S_Z Q_Z^{-1} Q_0 Q_Z^{-1} S_Z^* \\
&= \mathscr{L}_{A_Z} + \Pi_Z + M_Z.
\end{aligned}$$

(ii) Another straightforward calculation gives:

$$\begin{aligned}
\mathscr{R}'_Y(H) - \mathscr{R}'_Z(H) &= -S_Y Q_Y^{-1} (HB)^* - HBQ_Y^{-1} S_Y^* + S_Y Q_Y^{-1} \Sigma(H)^* \\
&\quad + \Sigma(H) Q_Y^{-1} S_Y^* + S_Y Q_Y^{-1} \Pi_2(H) Q_Y^{-1} S_Y^* \\
&\quad + S_Z Q_Z^{-1} (HB)^* + HBQ_Z^{-1} S_Z^* - S_Z Q_Z^{-1} \Sigma(H)^* \\
&\quad + \Sigma(H) Q_Z^{-1} S_Z^* - S_Z Q_Z^{-1} \Pi_2(H) Q_Z^{-1} S_Z^* \\
&= \left(S_Z Q_Z^{-1} - S_Y Q_Y^{-1} \right) (S_H - S_0)^* \\
&\quad + (S_H - S_0) \left(Q_Z^{-1} S_Z^* - Q_Y^{-1} S_Y^* \right) \\
&\quad + S_Y Q_Y^{-1} (Q_H - Q_0) Q_Y^{-1} S_Y^* \\
&\quad - S_Z Q_Z^{-1} (Q_H - Q_0) Q_Z^{-1} S_Z^* \\
&= M_Z - P_0 - (M_Y - P_0) \\
&\quad - \left(\Phi_Z(H) - S_H Q_H^{-1} S_H^* \right) + \Phi_Y(H) - S_H Q_H^{-1} S_H^* \\
&= M_Z - M_Y - \Phi_Z(H) + \Phi_Y(H).
\end{aligned}$$

(iii) This is immediate from (ii), since $\Phi_Y(Y) = 0$.

(iv) By (i) we have $\mathscr{R}(Y) = \mathscr{R}'_Y(Y) + M_Y$ and by (iii)

$$M_Y = -(\mathscr{R}'_Y - R'_Z)(Y) + M_Z - \Phi_Z(Y).$$

This gives the desired formula.

(v) If we subtract (iv) from (i), we get

$$\mathscr{R}(Z) - \mathscr{R}(Y) = \mathscr{R}'_Z(Z - Y) + \Phi_Z(Y),$$

which is (v). □

6. Newton's method applied to the equation $\mathscr{R}(X) = 0$

In this section, we derive a general non-local convergence result for the Newton algorithm applied to the rational matrix equation $\mathscr{R}(X) = 0$. For the special cases of deterministic algebraic Riccati equations this result can be found, e.g. in [15,16] for both CARE and DARE, and in the definite stochastic case special versions of the result were established in [6,24].

The method can be applied under the conditions that $\mathscr{R}$ is stabilizable and that the inequality $\mathscr{R}(X) \geqslant 0$ is solvable. Under these conditions, it will be shown that convergence takes place if the algorithm starts at any stabilizing initial matrix X_0.

Using Proposition 5.5(i), we can write the standard Newton-iteration for our problem in the following form:

$$X_{k+1} = X_k - \left(\mathscr{R}'_{X_k}\right)^{-1}(\mathscr{R}(X_k)) = -\left(\mathscr{R}'_{X_k}\right)^{-1}(M_{X_k}), \tag{29}$$

where $\mathscr{R}'_{X_k}$ is known from Proposition 5.2, and M_X was defined in (27). In each iteration step, the following linear matrix equation must be solved in order to obtain X_{k+1}:

$$A^*_{X_k}X + XA_{X_k} + \Pi_{X_k}(X) = M_{X_k}.$$

Theorem 6.1. *Assume that there exist a solution $\hat{X} \in \operatorname{dom}\mathscr{R}$ to $\mathscr{R}(X) \geqslant 0$ and a stabilizing matrix X_0. Then the iteration scheme* (29) *defines a sequence (X_k) in* $\operatorname{dom}\mathscr{R}$ *with the following properties*:
- (i) $\forall k = 1, 2, \ldots : X_k \geqslant X_{k+1} \geqslant \hat{X}$ *and* $\mathscr{R}(X_k) \leqslant 0$.
- (ii) $\forall k = 0, 1, 2, \ldots : \mathscr{R}'_{X_k}$ *is stable.*
- (iii) (X_k) *converges to a limit matrix* $X_\infty \in \operatorname{dom}\mathscr{R}$ *that satisfies* $\mathscr{R}(X_\infty) = 0$.
- (iv) X_∞ *is the greatest solution of* $\mathscr{R}(X) \geqslant 0$ *and* $\sigma(\mathscr{R}'_{X_\infty}) \subset \mathbb{C}_- \cup i\mathbb{R}$.

Proof. We prove (i) and (ii) inductively.

By assumption $\mathscr{R}'_{X_0}$ is stable, which settles the case $k = 0$.

Suppose that $X_0, \ldots, X_k$ have been constructed such that $\mathscr{R}'_{X_i}$ is stable for $i = 0, \ldots, k$, $X_1 \geqslant \ldots \geqslant X_k$ and $\mathscr{R}(X_i) \leqslant 0$ for $i = 1, \ldots, k$. Then X_{k+1} is well defined by (29) and satisfies

$$\mathscr{R}'_{X_k}(X_k - X_{k+1}) = \mathscr{R}(X_k). \tag{30}$$

We first prove $X_{k+1} \geqslant \hat{X}$. By the concavity of $\mathscr{R}$ (Proposition 5.5(v)) we have

$$\begin{aligned}\mathscr{R}'_{X_k}(\hat{X} - X_{k+1}) &= \mathscr{R}'_{X_k}(\hat{X} - X_k) + \mathscr{R}'_{X_k}(X_k - X_{k+1}) \\ &= \mathscr{R}'_{X_k}(\hat{X} - X_k) + \mathscr{R}(X_k) \overset{5.5(\mathrm{v})}{\geqslant} \mathscr{R}(\hat{X}) \geqslant 0. \end{aligned} \tag{31}$$

Since $\mathscr{R}'_{X_k}$ is stable, i.e. $-\mathscr{R}'_{X_k}$ is inverse positive by Corollary 3.8, we have $\hat{X} \leqslant X_{k+1}$ and hence $X_{k+1} \in \operatorname{dom} \mathscr{R}$.

By the same argument, if $k \geqslant 1$, it follows directly from (30) and $\mathscr{R}(X_k) \leqslant 0$ that $X_k - X_{k+1} \geqslant 0$. It remains to show that $\mathscr{R}'_{X_{k+1}}$ is stable and $\mathscr{R}(X_{k+1}) \leqslant 0$. Again by the concavity of $\mathscr{R}$ we have

$$0 \leqslant \mathscr{R}(\hat{X}) \leqslant \mathscr{R}(X_{k+1}) + \mathscr{R}'_{X_{k+1}}(\hat{X} - X_{k+1}). \tag{32}$$

From Proposition 5.5(iv) and (29) we have

$$\mathscr{R}(X_{k+1}) = \mathscr{R}'_{X_k}(X_{k+1}) + M_{X_k} - \Phi_{X_k}(X_{k+1}) = -\Phi_{X_k}(X_{k+1}) \leqslant 0, \tag{33}$$

which proves $\mathscr{R}(X_{k+1}) \leqslant 0$, and together with (32) we obtain

$$\mathscr{R}'_{X_{k+1}}(\hat{X} - X_{k+1}) \geqslant \Phi_{X_k}(X_{k+1}) \geqslant 0. \tag{34}$$

Now let us assume that $\mathscr{R}'_{X_{k+1}}$ is not stable. By Proposition 5.2 and Remark 3.3 both $\mathscr{R}'_{X_{k+1}}$ and its adjoint are resolvent positive; moreover, they have the same spectral abscissa. Thus, by Theorem 3.7, the instability of $\mathscr{R}'_{X_{k+1}}$ is equivalent to the following condition:

$$\exists V \in \mathscr{H}^n_+ \backslash \{0\}, \ \ \beta \geqslant 0: \quad \mathscr{R}'^{*}_{X_{k+1}}(V) = \beta V. \tag{35}$$

On the one hand, this implies

$$\left\langle V, \mathscr{R}'_{X_{k+1}}(\hat{X} - X_{k+1}) \right\rangle = \langle \beta V, \hat{X} - X_{k+1} \rangle \leqslant 0.$$

On the other hand, we have from (34)

$$\left\langle V, \mathscr{R}'_{X_{k+1}}(\hat{X} - X_{k+1}) \right\rangle \geqslant \langle V, \Phi_{X_k}(X_{k+1}) \rangle \geqslant 0.$$

Combined with the previous inequality, this gives $\langle V, \Phi_{X_k}(X_{k+1}) \rangle = 0$, and hence by Proposition 5.5(v)

$$\langle V, \mathscr{R}(X_{k+1}) - \mathscr{R}(X_k) \rangle = \left\langle V, \mathscr{R}'_{X_k}(X_{k+1} - X_k) \right\rangle,$$

whence by Lemma 4.3 $(\mathscr{R}'_{x_k})^*(V) = (\mathscr{R}'_{x_{k+1}})^*(V) = \beta V$, contradicting the stability of $\mathscr{R}'_{x_k}$. Thus, our assumption was wrong, and $\mathscr{R}'_{X_{k+1}}$ is stable. This concludes our proof of (i) and (ii) by induction.

(iii) By (i) the X_k converge to a matrix $X_\infty \in \operatorname{dom} \mathscr{R}$. Passing to the limit in (29) shows $\mathscr{R}(X_\infty) = 0$. Since $X_\infty \geqslant \hat{X}$ and $\hat{X}$ is an arbitrary solution of $\mathscr{R}(X) \geqslant 0$, the limit X_∞ is also the greatest solution of the inequality.

(iv) By (ii) all $\mathscr{R}'_{X_k}$ are stable; thus by continuity $\sigma(\mathscr{R}'_{X_\infty}) \subset \mathbb{C}_- \cup \mathrm{i}\mathbb{R}$. $\square$

Remark 6.2. Note that the Newton sequence of Theorem 6.1 is monotonic only after the first step. In fact, the first step might lead further away from X_∞.

From Theorem 6.1 we infer an existence theorem for the equation $\mathscr{R}(X) = 0$, which generalizes existence theorems for the definite CARE, DARE and STARE in LQ control theory. In [24], this result was given under the additional assumptions $\Sigma = 0$ and (A, P_0) detectable.

Corollary 6.3. *Assume* $Q_0 > 0$, $P_0 \geqslant S_0^* Q_0^{-1} S_0$, *and* $\mathscr{R}$ *is stabilizable. Then the equation* $\mathscr{R}(X) = 0$ *has a solution* $X \geqslant 0$.

Proof. The assumptions guarantee that $0 \in \operatorname{dom} \mathscr{R}$ and $\mathscr{R}(0) \geqslant 0$. Thus, Theorem 6.1 can be applied with $\hat{X} = 0$ and a stabilizing X_0. □

To apply the algorithm proposed in Theorem 6.1, it is crucial and in many cases not trivial to find a stabilizing matrix X_0. This problem is absent when $\mathscr{R}_0'$ is stable. The next theorem shows that if $P_0 \leqslant 0$, then the stability of $\mathscr{R}_0'$ is closely related to the existence of *negative definite* solutions of $\mathscr{R}(X) \geqslant 0$ and also to the stability of $\mathscr{L}_A + \Pi_1$.

Theorem 6.4. *Assume that* $P_0 \leqslant 0$, (A, P_0) *is observable, and the inequality* $\mathscr{R}(X) \geqslant 0$ *has a solution in* dom $\mathscr{R}$. *Then the following are equivalent*:

(i) $\mathscr{L}_A + \Pi_1$ *is stable.*
(ii) $\exists X \in \operatorname{dom} \mathscr{R} : \ \mathscr{R}(X) \geqslant 0$ *and* $X \leqslant 0$.
(iii) $X \in \operatorname{dom} \mathscr{R}$ *and* $\mathscr{R}(X) \geqslant 0 \ \Rightarrow \ X \leqslant 0$.
(iv) $X \in \operatorname{dom} \mathscr{R}$ *and* $\mathscr{R}(X) \geqslant 0 \ \Rightarrow \ X < 0$.

If additionally $Q_0 > 0$ *then all these statements are equivalent to*

(v) $\mathscr{R}_0'$ *is stable.*

The implications (i) ⇒ (iii) *and* (v) ⇒ (iii) (*as well as the trivial chain* (iv) ⇒ (iii) ⇒ (ii)) *hold without the observability assumption.*

Proof. (i) ⇒ (iii): On dom $\mathscr{R}$ the following inequality holds by definition

$$\begin{aligned}\mathscr{R}(X) &= P(X) - S(X)Q(X)^{-1}S(X)^* \\ &\leqslant P(X) = \mathscr{L}_A(X) + \Pi_1(X) + P_0.\end{aligned} \tag{36}$$

Thus, $\mathscr{R}(X) \geqslant 0$ implies $\mathscr{L}_A(X) + \Pi_1(X) \geqslant -P_0$, whence $X \leqslant 0$ if $\mathscr{L}_A + \Pi_1$ is stable (by Corollary 3.8).

(v) ⇒ (iii): By concavity

$$\mathscr{R}(X) \leqslant \mathscr{R}(0) + \mathscr{R}_0'(X) = P_0 - S_0 Q_0^{-1} S_0^* + \mathscr{R}_0'(X), \quad X \in \operatorname{dom} \mathscr{R}. \tag{37}$$

Thus, $Q(0) = Q_0 > 0$ and $\mathscr{R}(X) \geqslant 0$ imply $0 \in \operatorname{dom} \mathscr{R}$ and $\mathscr{R}_0'(X) \geqslant -P_0$, whence $X \leqslant 0$ if $\mathscr{R}_0'$ is stable.

To prove the remaining non-trivial implications (ii) ⇒ (i), (ii) ⇒ (v), and (ii) ⇒ (iv), we make use of the observability assumption.

Suppose that (A, P_0) is observable and $\mathscr{R}(X) \geqslant 0$ for some $X \in \operatorname{dom} \mathscr{R}$, $X \leqslant 0$. Then $\mathscr{L}_A(X) + \Pi_1(X) \geqslant -P_0$ by (36) and hence (iv) and (i) follow by Proposition 3.10. This proves (ii) ⇒ (i) and (ii) ⇒ (iv).

Finally, suppose that $Q_0 > 0$ and assume again that $\mathscr{R}(X) \geqslant 0$ for some $X \in \operatorname{dom} \mathscr{R}$, $X \leqslant 0$. Then $0 \in \operatorname{dom} \mathscr{R}$, and $\mathscr{R}_0'(X) = \mathscr{L}_{A_0}(X) + \Pi_0(X) \geqslant -P_0 + S_0 Q_0^{-1} S_0^* \geqslant 0$ by (37) (with $A_0 = A - BQ_0^{-1}S_0^*$). By Lemma 4.1 in [24] the pair $(A_0, -P_0 + S_0 Q_0^{-1} S_0^*)$ is observable, too, whence again by Proposition 3.10 $\mathscr{R}_0'(X)$ is stable and $X < 0$. This proves (ii) ⇒ (v) and (ii) ⇒ (iv) under the assumption $Q_0 > 0$. □

Note that, in the situation of Section 2, the stability of $\mathscr{L}_A + \Pi_1$ is equivalent to the internal stability of the given stochastic system (3) (by Remark 3.9).

7. Stabilizing solutions and quadratic convergence

By Theorem 6.1, there exists a greatest solution X_∞ to $\mathscr{R}(X) = 0$ and X_∞ is almost stabilizing, provided that $\mathscr{R}$ is stabilizable and the inequality $\mathscr{R}(X) \geqslant 0$ has a solution.

In many cases it is interesting to know, if there exists a *stabilizing* solution. For instance, we will see below, that this guarantees quadratic convergence of the Newton-iteration.

At first, we convince ourselves that if a stabilizing solution exists, it is also the greatest solution and thus unique. Then we will show that, for stabilizable $\mathscr{R}$ a stabilizing solution of $\mathscr{R}(X) = 0$ exists, if and only if the strict inequality $\mathscr{R}(X) > 0$ is solvable.

Lemma 7.1. *If for $Y, Z \in \operatorname{dom} \mathscr{R}$: $\mathscr{R}(Y) \leqslant \mathscr{R}(Z)$, and $\sigma(\mathscr{R}_Y') \subset \mathbb{C}_-$, then $Y \geqslant Z$.*

Proof. By concavity $\mathscr{R}_Y'(Y - Z) \leqslant \mathscr{R}(Y) - \mathscr{R}(Z) \leqslant 0$, whence $Y \geqslant Z$ by the stability of $\mathscr{R}_Y'$. □

Theorem 7.2. *The following are equivalent*:

(i) *$\mathscr{R}$ is stabilizable and $\exists \hat{X} \in \operatorname{dom} \mathscr{R} : \mathscr{R}(\hat{X}) > 0$.*

(ii) *There exists a stabilizing solution of the equation $\mathscr{R}(X) = 0$.*

Moreover, a stabilizing solution of the equation $\mathscr{R}(X) = 0$ is necessarily the greatest solution of the inequality $\mathscr{R}(X) \geqslant 0$ and thus unique; it coincides with X_∞ as given by Theorem 6.1.

Proof. (i) $\Rightarrow$ (ii): Let the sequence (X_k) be defined as in Theorem 6.1. Since $\mathscr{R}(\hat{X}) > 0$, inequality (31) holds in its strict form. Passing to the limit as $k \to \infty$ yields

$$\mathscr{R}'_{X_\infty}(\hat{X} - X_\infty) \geqslant \mathscr{R}(\hat{X}) > 0.$$

By continuity $\mathscr{R}'_{X_\infty}$ maps a whole neighbourhood of $\hat{X} - X_\infty \leqslant 0$ to $\operatorname{int}(\mathscr{H}^n_+)$. Thus, $\mathscr{R}'_{X_\infty}$ is stable by Corollary 3.8, i.e. X_∞ is a stabilizing solution of the equation $\mathscr{R}(X) = 0$.

If $\tilde{X}$ is another stabilizing solution of the equation $\mathscr{R}(X) = 0$, then by Lemma 7.1 $\tilde{X} \geqslant X_\infty$ and $X_\infty \geqslant \tilde{X}$, i.e. $X_\infty = \tilde{X}$.

(ii) $\Rightarrow$ (i): Let X_∞ be stabilizing. Then $\mathscr{R}'_{X_\infty}$ is a regular operator, and by the implicit function theorem, the equation $\mathscr{R}(X) - \epsilon I = 0$ is solvable for sufficiently small ϵ in a neighbourhood of X_∞ in $\mathscr{H}^n$. Hence (i). $\square$

We can show quadratic convergence of the sequence (X_k) defined in Theorem 6.1, provided there exists a stabilizing solution.

Theorem 7.3. *Assume that there exists a stabilizing solution $X \in \operatorname{dom} \mathscr{R}$ to $\mathscr{R}(X) = 0$. Let the sequence (X_k) and its limit X_∞ be given as in Theorem 6.1 (starting from an arbitrary stabilizing matrix $X_0 \in \operatorname{dom} \mathscr{R}$). Then there exists a constant κ such that*

$$\|X_{k+1} - X_\infty\| \leqslant \kappa \|X_k - X_\infty\|^2.$$

Proof. By concavity

$$\mathscr{R}'_{X_\infty}(X_\infty - X_{k+1}) \leqslant \mathscr{R}(X_\infty) - \mathscr{R}(X_{k+1}) \overset{\text{by (33)}}{=} \Phi_{X_k}(X_{k+1})$$

and since $-\mathscr{R}'_{X_\infty}$ is inverse positive

$$X_{k+1} - X_\infty \leqslant - \left(\mathscr{R}'_{X_\infty}\right)^{-1} (\Phi_{X_k}(X_{k+1})).$$

This gives the estimation

$$\|X_{k+1} - X_\infty\| \leqslant \left\| \left(\mathscr{R}'_{X_\infty}\right)^{-1} \right\| \|\Phi_{X_k}(X_{k+1})\|, \tag{38}$$

where the context tells whether $\|\cdot\|$ stands for the $\mathscr{H}^n$-norm or the induced operator norm. By (28)

$$\|\Phi_{X_k}(X_{k+1})\| \leqslant \left\| S_{X_{k+1}} Q^{-1}_{X_{k+1}} - S_{X_k} Q^{-1}_{X_k} \right\|^2 \|Q_{X_{k+1}}\|. \tag{39}$$

To estimate the first factor in this product we write

$$\begin{aligned} S_{X_{k+1}} Q^{-1}_{X_{k+1}} - S_{X_k} Q^{-1}_{X_k} &= (S_{X_{k+1}} - S_{X_k}) Q^{-1}_{X_{k+1}} \\ &\quad + S_{X_k} Q^{-1}_{X_{k+1}} (Q_{X_k} - Q_{X_{k+1}}) Q^{-1}_{X_k} \\ &= S'(X_{k+1} - X_k) Q^{-1}_{X_{k+1}} \\ &\quad - S_{X_k} Q^{-1}_{X_{k+1}} Q'(X_{k+1} - X_k) Q^{-1}_{X_k}. \end{aligned}$$

By the monotonicity of the X_k and the $Q_{X_k} > 0$ for $k \geqslant 1$ we have for $k \geqslant 1$:

$$\left\| Q_{X_k}^{-1} \right\| \leqslant \left\| Q_{X_\infty}^{-1} \right\|,$$
$$\|S_{X_k}\| \leqslant \|S_0\| + \|S'\| \max\{\|X_1\|, \|X_\infty\|\},$$

and thus for some $\kappa_0 \geqslant 0$:

$$\forall k \geqslant 1 : \left\| S_{X_{k+1}} Q_{X_{k+1}}^{-1} - S_{X_k} Q_{X_k}^{-1} \right\| \leqslant \kappa_0 \|X_{k+1} - X_k\|.$$

Using this in (38) and (39) we find a $\kappa > 0$ such that

$$\forall k \geqslant 1 : \|X_{k+1} - X_\infty\| \leqslant \kappa \|X_{k+1} - X_k\|^2 \leqslant \kappa \|X_k - X_\infty\|^2,$$

where the second inequality follows from the monotonicity of convergence. □

8. Monotonicity and concavity

In this section, we compare the greatest solutions of the matrix inequalities $\mathscr{R}^0(X) \geqslant 0$, $\mathscr{R}^1(X) \geqslant 0$, where $\mathscr{R}^0$ and $\mathscr{R}^1$ are of the type (19). Similar results for the algebraic Riccati equations CARE and DARE can be found e.g. in [15,17,22]. We will see that the linear matrix inequalities (LMIs) associated with $\mathscr{R}^i(X) \geqslant 0$ (see Remark 5.1) provide a convenient tool for the derivation of monotonicity results.

Proposition 8.1. *Given Λ, M, and Π as in* (17) *and* (18), *the following equivalences hold for $X \in \operatorname{dom} \mathscr{R}$*:

(i) $\mathscr{R}(X) \geqslant 0 \iff \Pi(X) + \Lambda(X) + M \geqslant 0.$

(ii) $\mathscr{R}(X) > 0 \iff \Pi(X) + \Lambda(X) + M > 0.$

Proof. It suffices to observe, that $\mathscr{R}(X)$ is the Schur complement of $\Pi(X) + \Lambda(X) + M$ with respect to the lower right block $Q(X)$, and $Q(X) > 0$ by definition for all $X \in \operatorname{dom} \mathscr{R}$. □

We will say that $\mathscr{R}$ is *the rational operator associated to Π, Λ*, and M. If Π and Λ are fixed, we will also write $\mathscr{R}^M$ for this operator to highlight its dependence on M.

For $i = 0, 1$, let $M_i \in \mathscr{H}^{n+\ell}$ and $\mathscr{R}^{M_i}$ be the rational matrix operators associated to Π, Λ, and M_i.

We first show that the greatest solution X_∞ depends monotonically on M.

Theorem 8.2. *Assume $M_1 \geqslant M_0$. If there exists a solution $X_0 \in \operatorname{dom} \mathscr{R}^{M_0}$ to $\mathscr{R}^{M_0}(X) = 0$ and $\mathscr{R}^{M_1}$ is stabilizable, then there exists a greatest solution X_1 to $\mathscr{R}^{M_1}(X) = 0$ and $X_1 \geqslant X_0$. If X_0 is stabilizing for $\mathscr{R}^{M_0}$, then X_1 is stabilizing for $\mathscr{R}^{M_1}$.*

Proof. By $M_1 \geqslant M_0$ and Proposition 8.1 we have $\operatorname{dom} \mathscr{R}^{M_0} \subset \operatorname{dom} \mathscr{R}^{M_1}$ and also $\mathscr{R}^{M_1}(X_0) \geqslant 0$. Thus by Theorem 6.1 there exists a greatest solution X_1 to $\mathscr{R}^{M_1}(X) = 0$ and $X_1 \geqslant X_0$.

If X_0 is stabilizing for $\mathscr{R}^{M_0}$, then by Theorem 7.2 there exists an $\tilde{X} \in \operatorname{dom} \mathscr{R}^{M_0}$ such that $\mathscr{R}^{M_0}(\tilde{X}) > 0$. Again by $M_1 \geqslant M_0$ and Proposition 8.1 we have $\mathscr{R}^{M_1}(\tilde{X}) > 0$. Thus, again by Theorem 7.2 X_1 is stabilizing. □

An analogous argument shows that X_∞ depends on M in a concave fashion (compare Definition 4.1).

Theorem 8.3. *Let $M_0, M_1 \in \mathscr{H}^{n+\ell}$ be arbitrary. For $\tau \in [0,1]$, set $M_\tau := (1-\tau)M_0 + \tau M_1$ and denote the rational operator associated to Π, Λ, and M_τ by $\mathscr{R}^{M_\tau}$.*

Assume that for $i = 0, 1$ there exist solutions $X_i \in \operatorname{dom} \mathscr{R}^{M_i}$ to $\mathscr{R}^{M_i}(X) = 0$ and that $\mathscr{R}^{M_{\tau_0}}$ is stabilizable for some $\tau_0 \in]0,1[$. Then there exists a greatest solution X_{τ_0} to $\mathscr{R}^{M_{\tau_0}}(X) = 0$ and $X_{\tau_0} \geqslant (1-\tau_0)X_0 + \tau_0 X_1$. If X_0 or X_1 is stabilizing, then so is X_{τ_0}.

Proof. Set $\hat{X}_{\tau_0} := (1-\tau_0)X_0 + \tau_0 X_1$. Obviously, $\hat{X}_{\tau_0} \in \operatorname{dom} \mathscr{R}^{M_{\tau_0}}$ and by Proposition 8.1

$$
\begin{aligned}
0 &\leqslant (1-\tau_0)(\Pi(X_0) + \Lambda(X_0) + M_0) + \tau(\Pi(X_1) + \Lambda(X_1) + M_1) \\
&= \Pi(\hat{X}_{\tau_0}) + \Lambda(\hat{X}_{\tau_0}) + M_{\tau_0},
\end{aligned} \tag{40}
$$

whence $\mathscr{R}^{M_{\tau_0}}(\hat{X}_{\tau_0}) \geqslant 0$. Thus by Theorem 6.1 there exists a greatest solution X_{τ_0} to $\mathscr{R}^{M_{\tau_0}}(X) = 0$ and $X_{\tau_0} \geqslant \hat{X}_{\tau_0}$.

If e.g. X_0 is stabilizing, then by Theorem 7.2 there exists an $\tilde{X}_0 \in \operatorname{dom} \mathscr{R}^{M_0}$ such that $\mathscr{R}^{M_0}(\tilde{X}_0) > 0$. Now we set $\tilde{X}_{\tau_0} := (1-\tau_0)\tilde{X}_0 + \tau_0 X_1 \in \operatorname{dom} \mathscr{R}^{M_{\tau_0}}$ and conclude as in (40), that $\Pi(\tilde{X}_{\tau_0}) + \Lambda(\tilde{X}_{\tau_0}) + M_{\tau_0} > 0$. Thus, again by Theorem 7.2, the greatest solution of the equation $\mathscr{R}^{M_{\tau_0}}(X) = 0$ is stabilizing. □

9. Continuity and analyticity

We use the notation from Section 8 and consider linear matrix inequalities of the form $R^M(X) := \Lambda(X) + \Pi(X) + M \geqslant 0$ with fixed Λ and Π but variable M. Our aim is to analyze the dependence of its greatest solutions on small variations of M. The associated rational matrix operator is denoted by $\mathscr{R}^M$. For convenience we introduce the following sets (depending on Λ and Π, but these operators are fixed):

$$
\begin{aligned}
\mathscr{M}_0 &:= \{M \in \mathscr{H}^{n+\ell} : \mathscr{R}^M \text{ is stabilizable}\}, \\
\mathscr{M}_1 &:= \{M \in \mathscr{H}^{n+\ell} : \mathscr{R}^M(X) = 0 \text{ is solvable in } \operatorname{dom} \mathscr{R}^M\}, \\
\mathscr{M}_2 &:= \{M \in \mathscr{H}^{n+\ell} : \Pi(X) + \Lambda(X) + M \geqslant 0 \\
&\qquad\qquad \text{has a greatest solution in } \mathscr{H}^n\}, \\
\mathscr{M}_3 &:= \{M \in \mathscr{H}^{n+\ell} : \mathscr{R}^M(X) = 0 \\
&\qquad\qquad \text{has a stabilizing solution in} \operatorname{dom} \mathscr{R}^M\}.
\end{aligned} \tag{41}
$$

The following identity and two inclusions follow from Theorem 6.1, the definitions in (41) and Theorem 8.2, respectively.

$$\mathscr{M}_3 = \mathscr{M}_0 \cap \left(\mathscr{M}_3 + \mathscr{H}_+^{n+\ell}\right) \subset \mathscr{M}_0 \cap \left(\mathscr{M}_1 + \mathscr{H}_+^{n+\ell}\right) \subset \mathscr{M}_2.$$

Our first theorem in this section shows that $\mathscr{M}_0$ and $\mathscr{M}_3$ are open and that on $\mathscr{M}_3$ the greatest solution of $\mathscr{R}^M(X) = 0$ depends analytically on M.

Theorem 9.1. *Given a positive linear map $\Pi : \mathscr{H}^n \to \mathscr{H}^{n+\ell}$ as in* (17) *and a linear map $\Lambda : \mathscr{H}^n \to \mathscr{H}^{n+\ell}$ as in* (18), *then the subsets $\mathscr{M}_0$, $\mathscr{M}_3$ are open in $\mathscr{H}^{n+\ell}$ and there exists a* (*real*) *analytic function $X_+ : \mathscr{M}_3 \to \mathscr{H}^n$ such that $X_+(M)$ is the stabilizing solution of $\mathscr{R}^M(X) = 0$ for all $M \in \mathscr{M}_3$.*

Proof. Partitioning every $M \in \mathscr{H}^{n+\ell}$ as in (18) we write

$$M = \begin{bmatrix} P_0^M & S_0^M \\ (S_0^M)^* & Q_0^M \end{bmatrix},$$

and for every $X \in \mathscr{H}^n$,

$$\begin{aligned} P_X^M &= A^*X + XA + \Pi_1(X) + P_0^M, \\ S_X^M &= XB + \Sigma(X) + S_0^M, \\ Q_X^M &= \Pi_2(X) + Q_0^M. \end{aligned}$$

If $X \in \operatorname{dom} \mathscr{R}^M = \{X \in \mathscr{H}^n \mid Q_X^M > 0\}$, we write

$$\Pi_X^M(H) = \begin{bmatrix} I \\ -(Q_X^M)^{-1}(S_X^M)^* \end{bmatrix}^* \Pi(H) \begin{bmatrix} I \\ -(Q_X^M)^{-1}(S_X^M)^* \end{bmatrix}^*,$$

$$A_X^M = A - B\left(Q_X^M\right)^{-1}\left(S_X^M\right)^*.$$

Clearly, $\mathscr{D} = \{(M, X) \in \mathscr{H}^{n+\ell} \times \mathscr{H}^n \mid Q_X^M > 0\}$ is non-empty and open in the real vector space $\mathscr{H}^{n+\ell} \times \mathscr{H}^n$ and the map $F : \mathscr{D} \to \mathscr{H}^n$ defined by

$$F : (M, X) \mapsto F(M, X) := \mathscr{R}_X^M = P_X^M - S_X^M\left(Q_X^M\right)^{-1}\left(S_X^M\right)^*, \quad (M, X) \in \mathscr{D},$$

is (real) analytic. As a consequence, the derivative

$$\frac{\partial F}{\partial X} : (M, X) \mapsto \frac{\partial F}{\partial X}(M, X) = (\mathscr{R}^M)'_X = \mathscr{L}_{A_X^M} + \Pi_X^M, \quad (M, X) \in \mathscr{D},$$

is an analytic map from $\mathscr{D}$ to $\mathscr{L}(\mathscr{H}^n)$.

Now let $M_0 \in \mathscr{M}_0$. Then there exists $X_0 \in \operatorname{dom} \mathscr{R}^{M_0}$ such that $\sigma((\mathscr{R}^{M_0})'_{X_0}) \subset \mathbb{C}_-$. Since the spectrum depends continuously on the operator, $(M_0, X_0) \in \mathscr{D}$ and $\partial F/\partial X$ is analytic on $\mathscr{D}$, there is an open ball $\mathscr{B}(M_0, \varepsilon) := \{M \in \mathscr{H}^{n+\ell} \mid |M -$

$M_0| < \varepsilon\}$ in $\mathscr{H}^{n+\ell}$ such that $X_0 \in \operatorname{dom} \mathscr{R}^M$ and $\sigma((\mathscr{R}^M)'_{X_0}) \subset \mathbb{C}_-$ for all $M \in \mathscr{B}(M_0, \varepsilon)$. Thus X_0 is stabilizing for all $\mathscr{R}^M$ with $M \in \mathscr{B}(M_0, \varepsilon)$, and this proves that $\mathscr{M}_0$ is open.

Now assume that $M_0 \in \mathscr{M}_3$ and let $X_0 \in \operatorname{dom} \mathscr{R}^{M_0}$ be the stabilizing solution of $\mathscr{R}^{M_0}(X) = 0$. Then $(M_0, X_0) \in \mathscr{D}$ and $\partial F/\partial X(M_0, X_0) = (\mathscr{R}^{M_0})'_{X_0}$ is stable, in particular invertible. As a consequence of the implicit function theorem for analytic functions [7], there is an open ball $\mathscr{B}(M_0, \varepsilon_0)$ in $\mathscr{H}^{n+\ell}$ such that for all $M \in \mathscr{B}(M_0, \varepsilon_0)$ there exists a unique solution $X(M) \in \mathscr{H}^n$ of $\mathscr{R}^M(X) = 0$ which depends analytically on $M \in \mathscr{B}(M_0, \varepsilon_0)$ and satisfies $X(M_0) = X_0$. But then $M \mapsto \partial F/\partial X(M, X(M)) = (\mathscr{R}^M)'_{X(M)}$ is continuous (even analytic) on $\mathscr{B}(M_0, \varepsilon_0)$ and since $\sigma((\mathscr{R}^{M_0})'_{X(M_0)}) \subset \mathbb{C}_-$ there exists $\varepsilon \in (0, \varepsilon_0)$ such that $\sigma((\mathscr{R}^M)'_{X(M)}) \subset \mathbb{C}_-$ for all $M \in \mathscr{B}(M_0, \varepsilon)$. Hence, $X(M)$ is a stabilizing solution of $\mathscr{R}^M(X) = 0$ for all $M \in \mathscr{B}(M_0, \varepsilon)$ and so $\mathscr{B}(M_0, \varepsilon) \subset \mathscr{M}_3$. This shows that $\mathscr{M}_3$ is open in $\mathscr{H}^{n+\ell}$, and the restriction of $X_+(\cdot)$ to $\mathscr{B}(M_0, \varepsilon)$ coincides with $X(\cdot)$ on $\mathscr{B}(M_0, \varepsilon)$. Therefore, $X_+(\cdot) : \mathscr{M}_3 \to \mathscr{H}^n$ is analytic. $\square$

Corollary 9.2. *Given Λ and Π as in Theorem* 9.1, *then the sets defined in* (41) *satisfy*

$$\mathscr{M}_3 \subset \mathscr{M}_0 \cap \mathscr{M}_1 \subset \overline{\mathscr{M}_3}.$$

Proof. The first inclusion follows directly from the definition. Now suppose $M_0 \in \mathscr{M}_0 \cap \mathscr{M}_1$ and $X_0 \in \operatorname{dom} \mathscr{R}^{M_0}$ is a solution of $\mathscr{R}^{M_0}(X) = 0$. By Theorem 9.1 we have $\mathscr{B}(M_0, \varepsilon_0) \subset \mathscr{M}_0$ for some $\varepsilon_0 > 0$. If we set $M_\varepsilon = M_0 + \varepsilon I$ for arbitrary $\varepsilon \in]0, \varepsilon_0[$, we get

$$R^{M_\varepsilon}(X_0) = \Lambda(X_0) + \Pi(X_0) + M_0 + \varepsilon I > 0.$$

It follows from Proposition 8.1(ii) and Theorem 7.2 that there exists a stabilizing solution of $R^{M_\varepsilon}(X) = 0$ and so $M_\varepsilon \in \mathscr{M}_3$. Since M_ε comes arbitrarily close to M_0 as $\varepsilon \to 0$, the corollary is proved. $\square$

In the situation of Section 2 it is of interest to examine, what happens, if γ decreases, i.e. if the attenuation value $\|L\|$ is approached. More generally, we can regard a decreasing convergent sequence of matrices $M_k \in \mathscr{M}_3$, $M_k \to \hat{M}$, and ask whether the corresponding greatest solutions $X_k := X_+(M_k)$ converge to a solution $\hat{X}$ of the equation $\mathscr{R}^{\hat{M}}(X) = 0$. If $\hat{M} \in \mathscr{M}_3$, this is clear from Theorem 9.1. But if $\hat{M} \in \partial \mathscr{M}_3$, it may happen that $\hat{X} = \lim_{k \to \infty} X_k$ exists and is the greatest solution of the LMI $\Pi(X) + \Lambda(X) + \hat{M} \geqslant 0$ but $\hat{X} \notin \operatorname{dom} \mathscr{R}_{\hat{M}}$, so that $\hat{M} \in \mathscr{M}_2 \backslash \mathscr{M}_3$. In the following, we will see that (under some controllability conditions) it suffices to assume $M_k \in \mathscr{M}_2$ for all k in order to obtain $\hat{M} \in \mathscr{M}_2$. Note that solutions of the LMI $R^M(X) := \Pi(X) + \Lambda(X) + M \geqslant 0$ satisfy $Q(X) \geqslant 0$ and $\operatorname{Ker} Q(X) \subset \operatorname{Ker} S(X)$, and may therefore be called generalized solutions of the associated rational inequality $\mathscr{R}^M(X) \geqslant 0$.

We will need the following two lemmas.

Lemma 9.3. *Let $(H_k)_{k\in\mathbb{N}}$ be an unbounded increasing sequence of Hermitian matrices in $\mathscr{H}^n$. Then there exists a nonzero vector $e \in \mathbb{K}^n$ such that* $\lim_{k\to\infty}\langle x, H_k x\rangle = \infty$ *for all $x \in \mathbb{K}^n$ with $\langle x, e\rangle \neq 0$.*

Proof. Replacing H_k by $H_k - H_0$ we may suppose without restriction of generality that $H_k \geqslant 0$. By a compactness argument we find a subsequence $(k_j)_{j\in\mathbb{N}}$ such that the limit $H = \lim_{j\to\infty} H_{k_j}/\|H_{k_j}\|$ exists. Since $\|H\| = 1$ there exists a unit eigenvector $e \in \mathbb{K}^n$, satisfying $He = e$. Every $x \in \mathbb{K}^n$ with $\langle x, e\rangle \neq 0$ can be written in the form $x = \alpha e + z$, where $\alpha \in \mathbb{K}\setminus\{0\}$ and $\langle z, e\rangle = 0$. Since $\langle Hz, e\rangle = 0$ it follows that

$$\lim_{j\to\infty}\left\langle x, \frac{H_{k_j}}{\|H_{k_j}\|}x\right\rangle = \langle x, Hx\rangle = \langle \alpha e + z, \alpha e + Hz\rangle = |\alpha|^2 + \langle z, Hz\rangle \geqslant |\alpha|^2,$$

whence $\langle x, H_{k_j}x\rangle \to \infty$ as $j \to \infty$, and so $\lim_{k\to\infty}\langle x, H_k x\rangle = \infty$ by monotonicity. $\square$

Lemma 9.4. *Suppose the Hermitian matrices*

$$H_k = \begin{bmatrix} A_k & B_k \\ B_k^* & C_k \end{bmatrix}$$

form an unbounded increasing sequence in $\mathscr{H}^{n+\ell}$ and assume that $\|C_k\| \leqslant \gamma$ for some $\gamma > 0$. Then, for arbitrary $x \in \mathbb{K}^n,\ u \in \mathbb{K}^\ell$,

$$\lim_{k\to\infty}\left\langle \begin{bmatrix} x \\ u \end{bmatrix}, H_k \begin{bmatrix} x \\ u \end{bmatrix}\right\rangle = \infty \iff \lim_{k\to\infty}\langle x, A_k x\rangle = \infty.$$

Proof. Again we may assume that $H_k \geqslant 0$, $k \in \mathbb{N}$. In order to prove "$\Rightarrow$", suppose that

$$\left\langle \begin{bmatrix} x \\ u \end{bmatrix}, H_k \begin{bmatrix} x \\ u \end{bmatrix}\right\rangle = \langle x, A_k x\rangle + 2\,\mathrm{Re}\langle x, B_k u\rangle + \langle u, C_k u\rangle \to \infty \quad \text{as } k \to \infty.$$

Since $H_k \geqslant 0$, we have by the Cauchy–Schwarz inequality (with respect to the semidefinite scalar product induced by H_k)

$$\langle x, A_k x\rangle\langle u, C_k u\rangle \geqslant |\langle x, B_k u\rangle|^2.$$

Hence $\langle x, A_k x\rangle \to \infty$ and "$\Rightarrow$" is proved. To show the converse implication, let us assume $a_k := \langle x, A_k x\rangle \to \infty$. Then

$$\left\langle \begin{bmatrix} x \\ u \end{bmatrix}, H_k \begin{bmatrix} x \\ u \end{bmatrix}\right\rangle \geqslant a_k^2 - 2|\langle x, B_k u\rangle| \geqslant a_k^2 - 2a_k\sqrt{\gamma}\|u\| \to \infty \quad \text{as } k \to \infty.$$

This concludes the proof. $\square$

By combining the preceding two lemmas we obtain:

Corollary 9.5. *If*

$$H_k = \begin{bmatrix} A_k & B_k \\ B_k^* & C_k \end{bmatrix} \in \mathscr{H}^{n+\ell}, \quad k \in \mathbb{N},$$

is an unbounded increasing sequence of Hermitian matrices with bounded lower right block C_k, then there exists a nonzero vector $e \in \mathbb{K}^n$, such that for all $x \in \mathbb{K}^n$ with $\langle e, x\rangle \neq 0$ and all $u \in \mathbb{K}^\ell$:

$$\lim_{k\to\infty} \left\langle \begin{bmatrix} x \\ u \end{bmatrix}, H_k \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle = \infty.$$

The next proposition is a mixed monotonicity/continuity result and is based on a controllability assumption for the pair (A, B).

Proposition 9.6. *Suppose Π and Λ are given as in Theorem* 9.1 *and $(M_k)_{k\in\mathbb{N}}$ is a bounded decreasing sequence in $\mathscr{M}_2$. For $k \in \mathbb{N}$, let X_k be the greatest solution to the LMI $R^{M_k}(X) = \Lambda(X) + \Pi(X) + M_k \geqslant 0$. If the pair (A, B) is controllable, then the X_k are bounded and converge to the greatest solution of $R^M(X) = \Lambda(X) + \Pi(X) + M \geqslant 0$, where $M = \lim_{k\to\infty} M_k$. In particular, $M \in \mathscr{M}_2$.*

Proof. Since the solution sets of the LMIs $R^{M_k}(\hat{X}) \geqslant 0$ are decreasing, so are the X_k. It is therefore enough to show, that $(X_k)_{k\in\mathbb{N}}$ is bounded, since then $\hat{X} = \lim_{k\to\infty} X_k$ exists and by continuity $R^M(\hat{X}) = \lim_{k\to\infty} R^{M_k}(X_k) \geqslant 0$, i.e. $\hat{X}$ solves the inequality $R^M(X) \geqslant 0$. Moreover, $\hat{X}$ is then the greatest solution of this inequality by monotonicity (see Theorem 8.2). We assume now that the sequence $(X_k)_{k\in\mathbb{N}}$ is *not* bounded and will show that this assumption leads to a contradiction.

Applying the Kalman–Yakubovich–Popov criterion to the inequality $\Lambda(X) + \Pi(X_k) + M_k \geqslant 0$ (e.g. [15,25]), we have for all k:

$$\forall \omega \in \mathbb{R} : \begin{bmatrix} (\mathrm{i}\omega I - A)^{-1} B \\ I \end{bmatrix}^* (\Pi(X_k) + M_k) \begin{bmatrix} (\mathrm{i}\omega I - A)^{-1} B \\ I \end{bmatrix} \geqslant 0. \tag{42}$$

There are two possibilities: Either $\Pi(X_k)$ is bounded below for $k \to \infty$ or not. In the first case, there exists (by monotonicity) $\hat{\Pi} := \lim_{k\to\infty} \Pi(X_k)$ and (42) remains valid, if we pass to the limit and replace $\Pi(X_k) + M_k$ by $\hat{\Pi} + M$. Thus, by the sufficiency of the Kalman–Yakubovich–Popov criterion, there exists a solution $\hat{X}$ of the inequality $\Lambda(X) + \hat{\Pi} + M \geqslant 0$. But since $\hat{\Pi} + M \leqslant \Pi(X_k) + M_k$ for all k, $\hat{X}$ also satisfies $R^{M_k}(X) \geqslant 0$. Hence X_k being the largest solution of this inequality, we have $\hat{X} \leqslant X_k$, i.e. $\hat{X}$ is a lower bound for all X_k, which contradicts our assumption.

So assume that $(\Pi(X_k))_{k\in\mathbb{N}}$ is not bounded. Since both $(M_k)_{k\in\mathbb{N}}$ and $(\Pi(X_k))_{k\in\mathbb{N}}$ are decreasing the sequence $(\Pi(X_k) + M_k)_{k\in\mathbb{N}}$ is decreasing and unbounded. We want to show that this is incompatible with (42). By (42) the lower right $\ell \times \ell$ block of $\Pi(X_k) + M_k$ is non-negative definite for all k, because $\lim_{\omega\to\infty} \|(\mathrm{i}\omega I -$

$A)^{-1}B\| = 0$. Thus the sequence $H_k := -\Pi(X_k) - M_k$ satisfies the assumption of Corollary 9.5, and there exists a nonzero vector $e \in \mathbb{K}^n$ such that for all $x \in \mathbb{K}^n$ with $\langle e, x\rangle \neq 0$ and all $u \in \mathbb{K}^\ell$:

$$\lim_{k\to\infty} \left\langle \begin{bmatrix} x \\ u \end{bmatrix}, (\Pi(X_k) + M_k) \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle = -\infty.$$

By the controllability of (A,B) there exist an $\omega_0 \in \mathbb{R}$ and a vector $u \in \mathbb{K}^\ell$ such that $\langle e, (\mathrm{i}\omega_0 I - A)^{-1}Bu\rangle \neq 0$. For if $e^*(\mathrm{i}\omega I - A)^{-1}B$ vanished identically in $\omega \in \mathbb{R}$, then all coefficients in the Laurent expansion $e^*(\mathrm{i}\omega I - A)^{-1}B = \sum_{k=0}^{\infty}(e^*A^kB)/((\mathrm{i}\omega)^{k+1})$ would vanish. Since $e \neq 0$ and $\mathrm{Im}(B, AB, \ldots, A^{n-1}B) = \mathbb{K}^n$ the latter is impossible.

For the constructed u we have

$$\lim_{k\to\infty} u^* \begin{bmatrix} (\mathrm{i}\omega_0 I - A)^{-1}B \\ I \end{bmatrix}^* (\Pi(X_k) + M_k) \begin{bmatrix} (\mathrm{i}\omega_0 I - A)^{-1}B \\ I \end{bmatrix} u = -\infty.$$

in contradiction to (42). $\square$

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